Retrospect

• Last lecture (Brown motion & Random walks)

• We have learned
  – Diffusion dominates material transport in \( \mu \text{m} \)-scales
  – Random walks & conformations of polymers
  – Diffusion ideas ==> permeability of membranes and the electrical potentials across them
  – Collective behavior of random motions can be predictable, even if individual motions are not.
  – Origin of friction: thermal motion induced collisions

• We'll outline: How viscous friction dominates mechanics in sub-cellular world?
Lecture 4. Life in the low lane: the low Reynolds-number world

Zhanchun Tu（涂展春）

Department of Physics, BNU

Email: tuzc@bnu.edu.cn

Homepage: www.tuzc.org
Main contents

• Friction in fluids
• Low Reynolds-number world
• Biological applications
§4.1 Friction in fluids
Suspension (悬浮液)

Gravity: $mg$  
Buoyant force: $\rho_w g V$

Net force: $(m - \rho_w V) g \equiv m_{net} g$

Profile of particle density

$$c(z) = c_0 e^{-\frac{m_{net} g z}{k_B T_r}}$$

(sedimentation (沉降) equilibrium in gravity)

Myoglobin (肌红蛋白): $m_{net} \approx 4 \text{ kg/mol}$

$$z_* \equiv \frac{k_B T_r}{m_{net} g} \approx 60 \text{ m}$$

In a 4cm test tube

$$\frac{c(z)}{c_0} = e^{-\frac{0.04}{60}} = 0.999$$

Thus the suspension never settles out (沉降). Called colloid
Centrifuge (离心机)

Centripetal (向心) acceleration: $\omega^2 r$

Equivalence principle

Centrifugal (离心) force $f = m \omega^2 r$

Centrifugal potential

$$U(r) = -\int f \, dr = -\frac{1}{2} m \omega^2 r^2$$

Boltzmann law $\Rightarrow$

$$c(r) = c_0 e^{m \omega^2 r^2 / 2 k_B T r}$$

(sedimentation equilibrium in centrifuge)

Problem: derive the above equation from Nernst-Planck formula
Sedimentation time scale (沉降系数)

- Sedimentation velocity

$$v_{drift} = \frac{m_{net} g}{\xi}$$

Depends on $g$
NOT intrinsic property of particles

- Sedimentation time scale

$$s = \frac{v_{drift}}{g} = \frac{m_{net}}{\xi}$$

unit: svedbergs ($10^{-13}$s)
Problem: Find the relation between $s$ and the mass of an ideal polymer

\[ s = \frac{m_{\text{net}}}{\xi} \quad \xi = 6\pi\eta_W R_g \]

\[ m_{\text{net}} = m - \rho_w V \]

\[ V \propto N : \text{number of monomers} \]

\[ \Rightarrow m_{\text{net}} \propto m \]

\[ R_g^2 = \left\langle \frac{1}{N} \sum_{n=1}^{N} (R_n - R_c)^2 \right\rangle = \left\langle \frac{1}{N} \sum_{n=1}^{N} \left( R_n - \frac{1}{N} \sum_{m=1}^{N} R_m \right)^2 \right\rangle \]

\[ = \frac{1}{2N^2} \sum_{n,m} \left\langle (R_n - R_m)^2 \right\rangle \]

Gaussian chain: \[ \left\langle (R_n - R_m)^2 \right\rangle = |n-m|b^2 \]

\[ \Rightarrow R_g^2 = Nb^2 / 6 \quad \text{for large } N \]

\[ \Rightarrow R_g \propto N^{1/2} \propto m^{1/2} \Rightarrow s \propto m^{0.5} \]

[Data: Meyerhoff & Schultz (1952)]
Hard to mix a viscous liquid

- Experiment: ink in corn syrup (玉米糊)
• Question:

2nd law==>mix. Is 2nd law violated in the above experiment?

Einstein relation \( \xi D = k_B T \)

Stokes relation \( \xi \propto \eta_{\text{corn}} \)

\[ D \propto \frac{k_B T}{\eta_{\text{corn}}} \]

\[ \eta \rightarrow \infty \Rightarrow D \rightarrow 0 \]

Thus the blob of ink cannot change much in a short time due to diffusion, and no evident mixing phenomenon.

Slowly Stirring (搅拌) causes an organized motion, in which successive layers of fluid simply slide over each other. Such a fluid motion is called laminar flow (层流).
§4.2 Low Reynolds-number world
Newtonian fluid

Uniform flow along $x$

Force applied on $y+dy$ layer by $y$ layer

$$\tau_f = -\eta \frac{dv}{dy}$$

Coefficient of viscosity

**Discussion**: guess the above formula from simple analysis.
Viscous critical force

• Dimensional analysis for Newtonian fluid

\[ [\eta]: \text{Pa s} = \text{kg m}^{-1} \text{ s}^{-1} \quad [\rho]: \text{kg m}^{-3} \]

We cannot construct a dimensionless quantity from \( \eta \) and \( \rho \).

Hence no intrinsic distinction between “thick” and “thin” fluids.

We cannot construct a dimension of length from \( \eta \) and \( \rho \).

Newtonian fluid “has no intrinsic length scale”, or is “scale invariant”.

Macroscopic experiments can tell us something relevant to a microscopic organism.
Viscous critical force

\[
\left[ \frac{\eta^2}{\rho} \right]: \frac{(\text{kg m}^{-1}\text{s}^{-1})^2}{\text{kg m}^{-3}} = \text{kg m s}^{-2} \quad \text{(dimension of force)}
\]

If an external force \( f < f_{\text{crit}} \), the fluid can be called “thick”. Friction will quickly damp out inertial effects. Flow is dominated by friction.

<table>
<thead>
<tr>
<th>Fluid</th>
<th>( \rho_m ) (kg m(^{-3}))</th>
<th>\eta (Pa s)</th>
<th>( f_{\text{crit}} ) (N)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air</td>
<td>1</td>
<td>2 \times 10^{-5}</td>
<td>4 \times 10^{-10}</td>
</tr>
<tr>
<td>Water</td>
<td>1000</td>
<td>0.0009</td>
<td>8 \times 10^{-10}</td>
</tr>
<tr>
<td>Corn syrup</td>
<td>1000</td>
<td>5</td>
<td>0.03</td>
</tr>
</tbody>
</table>

Typical scale of forces inside cells is in pN range.

Friction rules the world of the inner cell!
Reynolds number

A ball moves in water

• Dimensional analysis

\[ \eta: \text{Pa} \text{ s} = \text{kg} \text{ m}^{-1} \text{ s}^{-1} \quad \rho: \text{kg} \text{ m}^{-3} \]

\[ R: \text{m} \quad v: \text{m s}^{-1} \]

\[ \left[ \frac{\rho R v}{\eta} \right]: \text{Dimensionless number} \]

• Reynolds number

\[ \mathcal{R} \equiv \frac{\rho R v}{\eta} \]

[Reynolds (1880s)]
• Physical meaning I

Time for a ball moving $2R$: $t = \frac{2R}{v}$

The same volume of water moves $2R$ in time $t$, the mean acceleration is estimated as

$$\frac{1}{2}at^2 = 2R \Rightarrow a = \frac{4R}{t^2} = \frac{v^2}{R}$$

inertial term of water = mass $\times$ acceleration $\sim \rho R^3 \left( \frac{v^2}{R} \right) = \rho R^2 v^2$

friction between the ball and water: $f_{frict} = \xi v = 6\pi \eta R v \sim \eta R v$

\[
\frac{\text{inertial term}}{\text{friction}} = \frac{\rho Rv}{\eta} = \mathcal{R}
\]

The inertial term can be safely omitted if $\mathcal{R} \ll 1$
Physical meaning II

- Kinetic energy of water
  \[ E_K \sim \rho R^3 v^2 \]

- Friction between the ball and water
  \[ f_{\text{frict}} = \xi v = 6\pi \eta R v \sim \eta R v \]

- Dissipative work done by the friction in displacement \( R \):
  \[ W_{\text{frict}} = f_{\text{frict}} R \sim \eta R^2 v \]

\[ \frac{E_K}{W_{\text{frict}}} = \frac{\rho Rv}{\eta} = \mathcal{R} \]
• Physical meaning III

\[
\begin{align*}
  f_{\text{frict}} &\sim \eta R v \\
  f_{\text{crit}} &\approx \eta^2 / \rho \\
  \frac{f_{\text{frict}}}{f_{\text{crit}}} &\sim \frac{\rho R v}{\eta} = \mathcal{R}
\end{align*}
\]

\[
\mathcal{R} \ll 1 \Rightarrow f_{\text{frict}} = f_{\text{ext}} \quad \text{(omit the inertial term)}
\]

Force applied on water by the ball, also=the external pulling force on the ball

Thus

\[
f_{\text{ext}} \ll f_{\text{crit}} \quad \text{for} \quad \mathcal{R} \ll 1
\]
• Problem: Calculate the Reynolds numbers
  – (1) A 30 m whale, swimming in water at 10m/s
  – (2) A 1 μm bacterium, swimming at 30 μm/s

Solutions:

(1) \[ \Re = \frac{\rho R v}{\eta} = \frac{10^3 \times 30 \times 10}{10^{-3}} = 3 \times 10^8 \gg 1 \]

(2) \[ \Re = \frac{\rho R v}{\eta} = \frac{10^3 \times 10^{-6} \times (30 \times 10^{-6})}{10^{-3}} = 3 \times 10^{-5} \ll 1 \]

Bacteria live in the world of low Reynolds number where the viscous friction rules their motions!
Time reversal properties

- Time reversal operation \((t \rightarrow -t)\)
Example I: falling ball in gravity

Dynamical equation
\[ \frac{d^2 z}{dt^2} = -g \]

Equation of motion
\[ z(t) = v_0 t - \frac{1}{2} g t^2 \]

Time reversal
Dynamical equation
\[ \frac{d^2 z}{dt^2} = -g \]

Equation of motion
\[ z_r(t) = z(-t) = -v_0 t - \frac{1}{2} g t^2 \]

Time reversal Invariance!
• Example II: diffusion equation

\[
\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2}
\]

solution

\[
c(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/4Dt}
\]

\[
\frac{-\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2}
\]

NOT a solution

\[
c_r(x,t) = c(x,-t)
\]

NOT time-reversal invariant

\[
= \frac{1}{i\sqrt{4\piDt}} e^{x^2/4Dt}
\]
• Viscous friction rule is **not** time-reversal invariant

A ball in highly viscous fluid

Dynamical equation

\[
\xi \frac{dx}{dt} = f
\]

NOT time-reversal invariant

Time reversal dynamical equation

\[
-\xi \frac{dx}{dt} = f
\]

Something irreversible in the motion ruled by friction!

**Discussion:** Does this conclusion contradict the experiment of ink drop in corn syrup?
• Newtonian fluid & simple elastic solid

Simple elastic solid

\[ \tau = -G \frac{du}{dy} \]  
(Hooke relation)

Time-reversal invariant!

Newtonian fluid

\[ \tau_f = -\eta \frac{dv}{dy} = -\eta \frac{d}{dy} \left( \frac{du}{dt} \right) \]

NOT time-reversal invariant!
§4.3 Biological applications
Swimming of microorganisms

- Strictly reciprocating (往复) motion

- The paddles (桨) move backward in speed \( v \) relative to the body ==> the forward motion of the body in speed \( u \) relative to water.

- The paddles move forward in speed \( v' \) relative to the body ==> the backward motion of the body in speed \( u' \) relative to water.

- Repeating a.
Friction force of body from water:

\[ f_b = -\xi_0 u \]

Speed of paddles relative to water:

\[ u_p = -v + u \]

Friction force of paddles from water:

\[ f_p = -\xi_1 u_p = -\xi_1 (u - v) \]

Force balance:

\[ f_b + f_p = 0 \Rightarrow u = \xi_1 v / (\xi_0 + \xi_1) \]

Displacement of body after time \( t \):

\[ \Delta x = ut = \xi_1 vt / (\xi_0 + \xi_1) \]
Friction force of body from water:

\[ f_b' = -\xi_0 (-u') = \xi_0 u' \]

Speed of paddles relative to water:

\[ u_p' = v' + (-u') = v' - u' \]

Friction force of paddles from water:

\[ f_p' = -\xi_1 u_p' = -\xi_1 (v' - u') \]

Force balance:

\[ f_b' + f_p' = 0 \Rightarrow u' = \xi_1 v' / (\xi_0 + \xi_1) \]

Displacement of body after time \( t' \):

\[ \Delta x' = -u' t' = -\xi_1 v' t' / (\xi_0 + \xi_1) \]
Constraint of relative speed in a period $t+t'$:

the paddles should return to their original positions on the body

\[ vt - v' t' = 0 \]

Net displacement of the body in a period $t+t'$:

\[ \Delta x + \Delta x' = \xi_1 \frac{vt}{(\xi_0 + \xi_1)} - \xi_1 \frac{v' t' }{(\xi_0 + \xi_1)} \]

\[ = \xi_1 \left( \frac{vt - v' t' }{(\xi_0 + \xi_1)} \right) = 0 \]


Question: What other options does a microorganism have?

The required motion must be periodic, so that it can be repeated. It can’t be of the reciprocal type described above.
• Ciliary propulsion (纤毛推进)

Many cells use cilia to generate net thrust (推力). Each cilium contains internal filaments and motors which create an overall bend in the cilium.

Typical motion of cilia.

Periodic but NOT reciprocal!
Understand intuitively the trick to generate the net motion

Quickly retract paddles and then quickly extend paddles.

Key: viscous friction coefficient of paddles are proportional to their lengths.

Net displacement: \[ \frac{\xi_1}{\xi_0 + \xi_1} - \frac{\xi_1'}{\xi_0 + \xi_1'} \] \(\times vt\)

\[ = \xi_0 vt \frac{\xi_1 - \xi_1'}{\xi_0 + \xi_1}(\xi_0 + \xi_1') > 0 \]
- **Bacterial flagella**
Mechanism of flagellar propulsion (鞭毛推进机理)

Key: although \( v \) is perpendicular to \( z \), \( df \) can have \( z \) component. Why?

\[
\begin{align*}
\mathbf{f}_{\parallel} &= -\zeta_\parallel v_{\parallel} \\
\mathbf{f}_{\perp} &= -\zeta_\perp v_{\perp} \\
\end{align*}
\]

Generally, \( \xi_\perp > \xi_\parallel \)

\[ \Rightarrow \mathbf{f} = \mathbf{f}_{\perp} + \mathbf{f}_{\parallel} \]

not antiparallel \( v \)

Discussion: prove that for a long flagellum

\[
(\int d\mathbf{f})_{\parallel z}
\]
Discussion: There exists torsion around $z$ although the total force along $z$. How can bacteria resist rotating themselves around $z$ axis?

Option 1: the bacteria are rough enough (i.e., large surface area), such that

$$\xi_{rotation} \text{ is very large } \Rightarrow \omega_{bacteria} = \frac{\text{torsion}}{\xi_{rotation}} \to 0$$

Option 2: the bacteria have even number of flagella. A pair of flagella have the opposite rotation. The torsion is canceled out while force along $z$ axis is double.

Option 3: ...... (maybe)
• Dimer (二聚体) of two reciprocating motion

Single pair of paddles

\[ \text{Strictly reciprocal} \]
\[ \Rightarrow \text{No net motion} \]

Dimer of 2 pairs of paddles

\[ \text{Each pair: strictly reciprocal} \]
\[ \text{Dimer: nonreciprocal} \]
\[ \Rightarrow \text{Net motion is possible} \]

No many-scallop theorem [Lauga and Bartolo (2008) PRE]
Blood flow in capillary (毛细血管)

• Model

\[ \mathcal{R} = \frac{\rho R v}{\eta} = \frac{10^3 \times 10^{-5} \times 0.1}{10^{-3}} = 1 \]

Reynolds's study suggests laminar pipe flow for \( \mathcal{R} < 10^3 \)

Constraints:

\[ v(R) = 0 \quad (\text{non-slip}) \]
\[ v(0) < \infty \]
• **Force balance equation**

**Pressure** contributes a force: \[ df_p = (2\pi r \, dr) \, p \]

**Viscous force** from inner layer fluid **drags forward** the fluid shell:

\[ df_{\text{in}} = -\eta [2\pi r \, L] \frac{dv(r)}{dr} \quad \text{note: } \frac{dv}{dr} < 0 \]

**Viscous force** from outer layer fluid **pulls backward** the fluid shell:

\[ df_{\text{out}} = \eta [2\pi (r + dr) \, L] \frac{dv(r')}{{dr'}} \bigg|_{r' = r + dr} \]

\[ = \eta [2\pi (r + dr) \, L] \left[ \frac{dv(r)}{dr} + \frac{d^2 v(r)}{dr^2} \, dr \right] \]

\[ df_p + df_{\text{in}} + df_{\text{out}} = 0 \Rightarrow \frac{rp}{\eta L} + \frac{dv}{dr} + r \frac{d^2 v}{dr^2} = 0 \]
• Flow profile and flow rate

Problem: prove that the general solution is

\[ v(r) = A + B \ln r - r^2 \frac{p}{4 \eta L} \]

Problem: prove that

\[ B = 0 \quad \text{and} \quad A = \frac{pR^2}{4 \eta L} \]

Flow profile

\[ v(r) = \frac{p (R^2 - r^2)}{4 \eta L} \]

Flow rate (Hagen–Poiseuille relation)

\[ Q = \int v \, dA = 2\pi \int v(r) \, r \, dr = \frac{\pi R^4 p}{8 \eta L} \propto R^4 \]

Blood vessels can efficiently regulate flow with only slight dilations or contractions!
Viscous force at DNA replication fork

- **Question on DNA replication**

Since the two single strands cannot pass through each other, the original must continually rotate (arrow). Would frictional force resisting this rotation be enormous?
Estimate the effect of friction

Constraints:

\[ v(R) = \omega R \quad \text{and} \quad v(\infty) = 0 \]

Torsion from the fluid inner \( r \) that drags the fluid shell rotating counter-clockwise:

\[ T_{in} = -\eta \frac{dv(r)}{dr} (2\pi r L) r = -2\pi \eta L r^2 \frac{dv(r)}{dr} \]

Note: \( \frac{dv}{dr} < 0 \)

Torsion from the fluid inner \( r + dr \) that drags the fluid shell rotating clockwise:

\[ T_{out} = 2\pi L \eta (r + dr)^2 \frac{dv(r')}{dr'} \bigg|_{r' = r + dr} \]

\[ T_{in} + T_{out} = 0 \Rightarrow d \left( r^2 \frac{dv}{dr} \right) = 0 \quad \Rightarrow v(r) = \frac{\omega R^2}{r} \quad \text{and} \quad \frac{dv}{dr} = -\frac{\omega R^2}{r^2} \]
Friction torsion between the rod and fluid:

\[ T(R) = -2\pi \eta L \left( r^2 \frac{dv(r)}{dr} \right) \bigg|_{r=R} = 2\pi \eta LR^2 \omega \]

\[ \Rightarrow \text{Book: } 4\pi, \text{ which is correct?} \]

The rod rotates \( 2\pi \) for opening each helical turn. The work of friction

\[ W_{frict} = T(R) \times 2\pi = 4\pi^2 \eta LR^2 \omega \]

DNA polymerase synthesizes new DNA in \( E. \ coli \) at a rate \( \sim 1000 \text{bp/s} \)

\[ \omega = \frac{2\pi}{\text{turn}} \times \frac{1000 \text{bp/s}}{10.5 \text{bp/turn}} \approx 600 \text{rad/s} \quad \text{Problem: please estimate } R. \]

\[ \Rightarrow W_{frict} = 4\pi^2 \eta LR^2 \omega \approx 2.4 \times 10^{-22} \text{J} = 0.06 \ k_B \ T_r \]
DNA helicase (解旋酶)

Helicase walks along the DNA in front of the polymerase, unzipping the double helix as it goes along.

Energy source: ATP.

\[
1 \text{ ATP} \equiv 20 k_B T_r \quad \text{(in physiological condition)}
\]

\[
W_{frict} = 0.06 k_B T_r \ll 20 k_B T_r
\]

The energy lost of viscous friction can be negligible!
Cytoplasmic streaming (胞浆流动)

Chara corallina
珊瑚轮藻

leaf 1. 鳞片
node 2. 叶柄
internodal cell 3. 嫖间细胞

an internodal cell
close-up of indifferent zone (3D view)

4. chloroplasts
5. indifferent zone
6. endoplasm
7. vacuole

[PNAS 105(2008) 3663]
Intracellular (细胞内) material transport: Flow dominate or diffusion dominate?

Characteristic length: $L$  
Characteristic velocity of flow: $U$

Diffusion constant: $D$

$\langle x^2 \rangle = 2D \cdot t \Rightarrow$ Time for molecules diffusing $L$: 
$t_D = \frac{L^2}{2D} \approx \frac{L^2}{D}$

Time for molecules reach $L$ due to flow: 
$t_f = \frac{L}{U}$

Peclet number: $P_e = \frac{t_D}{t_f} = \frac{UL}{D}$

If $P_e \gg 1$, molecules reach $L$ more quickly due to flow than diffusion. Thus, flow dominates the transport.

If $P_e \ll 1$, then diffusion dominates the transport.

In an internodal cell: $L=1 \text{mm}$, $U=30 \mu\text{m/s}$, $D=10^{-5} \text{cm}^2/\text{s}$.

$P_e = \frac{UL}{D} = 30 \gg 1 \Rightarrow$ flow dominates the transport.
• Physical description

Flow of streaming: \( \eta \nabla^2 u = \nabla p, \quad \nabla \cdot u = 0 \) (stokes equation)

\[ \nabla \cdot u = 0 \]

Fluid velocity

The change of concentration of materials:

\[ c_t + u \cdot \nabla c = D \nabla^2 c \] (advection-diffusion equation)

Note: if you stand a frame S' with the velocity of \( u \), we will observe

\[ c_t' = D \nabla' \nabla' c \]

For large Peclet number, diffusion can be neglected, thus

\[ c_t + u \cdot \nabla c = 0 \] (advection equation)

If only we have solved the stokes equation and obtain \( u(x,y,z,t) \), then we can obtain \( c(x,y,z,t) \).
**Spiral flow**

\[ u = v(r, \varphi)H + J(r, \varphi) \]

Parallel to spiral

In radial direction

Flow stems from surface, then passes the inner (vacuole), and reaches the other points in surface again. Helpful to material transport across the vacuole.

Max at pitch/R=3

This ratio corresponds to max nutrient (营 养) uptake rate from environment.
§ Summary & further reading
Summary

- Suspension
  - “Suspension never settles out (沉降)”
  - Sedimentation time scale: \( s = \frac{m_{\text{net}}}{\xi} \)

- Viscous liquid
  - Hard to mix a viscous liquid
  - Friction rules the world of the cell!
  - Critical viscous force: \( f_{\text{crit}} = \frac{\eta^2}{\rho} \)

- Reynolds number: \( \mathcal{R} = \frac{\rho R v}{\eta} \)
  - inertial term / friction
  - kinetic energy/dissipative work of friction
  - external force/critical viscous force
• Time reversal properties
  – 2nd Newtonian law: time-reversal invariant
  – Viscous friction rule: NOT time-reversal invariant
  – Simple elastic solid: time-reversal invariant
  – Viscous fluid: NOT time-reversal invariant

• Swimming of microorganisms:
  – Strictly reciprocating motion won’t work
  – Periodic but NOT reciprocal motion can work: Ciliary propulsion & flagellar propulsion
• Blood flow in capillary
  - Flow rate: \[ Q = \frac{\pi R^4 p}{8 \eta L} \]
  - Blood vessels can efficiently regulate flow with only slight dilations or contractions

• DNA replication fork
  - Dissipative work of friction: \[ W_{frict} = 4 \pi^2 \eta LR^2 \omega \approx 0.06 k_B T_r \]
  - Energy lost of viscous friction can be negligible: \[ W_{frict} \ll E_{ATP} \approx 20 k_B T_r \]

• Cytoplasmic streaming
  - Peclet number: \[ P_e = \frac{UL}{D} \]
  - Physical description: \[ \eta \nabla^2 u = \nabla p, \ \nabla \cdot u = 0 \]
  \[ c_t + u \cdot \nabla c = D \nabla^2 c \]
Further reading

