Hardcore bosons on the dual of the bowtie lattice

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We study the zero temperature phase diagram of hardcore bosons on the dual of the bowtie lattice. Two types of striped diagonal long-range order (striped order I and striped order II) are discussed. A state with type-II striped order and superfluidity is found, even without nearest-neighbor repulsion. The emergence of such a state is due to the inhomogeneity and the anisotropy of the lattice structure. However, neither the translational symmetry nor the symmetry between sublattices of the original lattice is broken in this state. In this paper, we restrict a 'solid state' of lattice bosons as a diagonal long-range ordered state breaking either the translational symmetry of the original lattice or the symmetry of different sublattices. We thus name such a phase a striped superfluid phase. In the presence of a nearest-neighbor repulsion, we find two striped charge density wave phases with boson density ρ = 1/2 (with striped order I) and ρ = 2/3 (with striped order II) respectively, when the hopping amplitude is small compared with the repulsion. The striped charge density wave I state is a solid, in which the translational symmetry of the original lattice is broken. We observe a rather special first-order phase transition showing an interesting multi-loop hysteretic phenomenon between the two striped charge density wave phases when the hopping term is small enough. This can be accounted for by the special degeneracy of the ground states near the classical limit. The striped superfluid phase re-appears outside the two striped charge density wave phases. The transition between the striped charge density wave I and striped superfluid phases is first order, while the transition between striped charge density wave II and the striped superfluid phases is continuous. We find that the superfluid stiffness is anisotropic in the striped superfluid states with and without repulsion. In the striped superfluid state with repulsion, the superfluid stiffness is subject to different types of anisotropy in the region near half filling and above 2/3-filling.

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I. INTRODUCTION

The supersolid, a novel quantum state with simultaneous diagonal long-range (density) order (DLRO) and off-diagonal long-range (superfluid) order (ODLRO), was introduced nearly half a century ago. This exotic quantum state has attracted considerable research interests in recent years. Owing to the fast development of laser cooling techniques, various optical lattices can be realized, which makes the investigation of supersolidity of bosons on discrete lattices more realistic. For softcore bosons, the supersolid phase emerges by the "defect condensation" mechanism, where doped bosons (holes) act as interstitials (vacancies) in the crystal. However, for hard-core bosons with nearest-neighbor repulsion, no supersolidity was found on the square, honeycomb, kagome, star lattice, and Shastry-Sutherland lattices, due to the instability of such a phase on these lattices, which leads to phase separation into a pure solid and a superfluid for all values of the interaction strength. The triangular lattice is an exception, on which supersolidity was found for hardcore bosons. The extensive degeneracy of the classical ground states at half filling is lifted by quantum fluctuations and the ground state attains DLRO and ODLRO simultaneously, thus forming a supersolid. Moreover, it was suggested that frustration induced by next-nearest neighbor interactions on the square lattice can also lead to supersolidity. Although frustration alone may not be enough to produce a supersolid, these findings show a different route to supersolidity, which is based on an order-by-disorder mechanism, by which a quantum system avoids classical frustration. However, these are the only examples to show evidence of supersolid for hardcore bosons and all the investigations considered simple homogeneous lattices. It is natural to investigate the behavior of bosons also on inhomogeneous lattices. As we shall show later in this paper, the hard-core bosons on such a lattice can have macroscopic degeneracy of classical ground states without geometrical frustrations present and other interesting properties that are worthwhile to be investigated.

Let us consider hardcore bosons with nearest-neighbor repulsion on the dual of the bowtie lattice, which is anisotropic in two directions and consists of two types of sites with different coordination number, see Fig. 1. The corresponding Bravais lattice is a rectangular lattice. It has a unit cell of 3 sites A, B, and C with the primitive vectors \( \mathbf{a}_1, \mathbf{a}_2 \) as shown in Fig. 1. Although the lattice looks rather complex and weird, and there is no report of such a crystal structure exists in nature to our knowledge, it is quite common that a material has a crystal structure formed by attaching a basis of atoms to each Bravais lattice point, and thus bears the inhomogeneity and the anisotropy that we are looking at. Moreover, the corresponding optical lattice can be realized experimentally in principle by directly projecting a phase hologram that contains the lattice structure onto the atom plane.
The Hamiltonian is
\[ H = -t \sum_{\langle i,j \rangle} (a_i^\dagger a_j + a_j^\dagger a_i) + V \sum_{\langle i,j \rangle} n_in_j - \mu \sum_i n_i, \] (1)

where \( a_i^\dagger (a_i) \) creates (annihilates) a boson at site \( i \), \( t \) is the nearest-neighbor hopping amplitude, \( V \) the nearest-neighbor repulsion, \( \mu \) the chemical potential, and \( n_i = 0 \) or \( 1 \).

The model can be mapped onto the spin-1/2 XXZ model on the same lattice in the usual way,
\[ H_s = -J \sum_{\langle i,j \rangle} (S_i^+ S_j^- + S_i^- S_j^+) + J_z \sum_{\langle i,j \rangle} S_i^z S_j^z - \sum_i h(i) S_i^z, \] (2)

where \( J = 2t \) is the in-plane exchange, \( J_z = V \) is the exchange in the \( z \) direction, and \( h(i) \) is a staggered external magnetic field given by \( h(i) = \mu - 2V \) for sites with coordination number 4 and \( \mu - 3V/2 \), otherwise. This marks the special character of the model. The solid state of the bosons is equivalent to spontaneous magnetic order in the \( z \) direction.

This paper is organized as follows: We first discuss the classical limit of the model at zero temperature in Sec. II. We show that, even in this simple limit, the model shows interesting properties, which are introduced by the inhomogeneity and the anisotropy of the lattice. Then in Sec. III, we present quantum Monte Carlo simulation results for the noninteracting \((V = 0)\) and the interacting \((V \neq 0)\) cases, focusing on the full phase diagram in both cases and in particular the novel phase with striped order and superfluidity simultaneously. We find that the superfluid stiffness is anisotropic in the striped superfluid states with and without repulsion. For the interacting system, we report a rather special first-order phase transition showing an interesting multi-loop hysteresis phenomenon between two striped charge density wave phases when the hopping term is small enough. We conclude in Sec. IV with some discussions.

II. CLASSICAL LIMIT

In the classical limit \((t = 0)\), at zero temperature, there are only two phases if the nearest-neighbor repulsion (or the exchange in \( z \) direction) \( V \) is absent. The lattice is empty when \( \mu < 0 \), full if \( \mu > 0 \).

With the repulsion \( V \) present, there exist four phases. For \( \mu/V < 0 \), the lattice is empty. For \( \mu/V > 4 \), the lattice is full. In the region \( 0 < \mu/V < 2 \), the lattice is half filled. The model shows a solid ordering (antiferromagnetic order in \( z \) direction) which breaks the translational symmetry of the original lattice, as shown in Fig. 1(a). Two unit cells half filled with three bosons form a new unit cell of a solid at a wavevector \((\pi,0)\), showing the character of a striped solid or a striped charge density wave (SCDW), which is analogous to the period-two striped state at a wavevector \((\pi,\pi/\sqrt{3})\) found for hardcore bosons on the triangular lattice with nearest and next-nearest neighbor repulsion\(^{20}\). We refer to this order as striped order I. In the region \( 2 < \mu/V < 4 \) the lattice is \( 2/3 \) filled, showing a DLRO again. Two of three sites in a unit cell are filled and form the unit cell of a phase with a wavevector \((2\pi,0)\), as shown in Fig. 1(b). However, this ordering does not break the translational symmetries of the original Bravais lattice, nor the symmetry of A and B sublattices. (The spontaneously breaking of the symmetry between two sublattices has been reported in Ref. 10). Thus, we would not say the bosons form a solid. In this paper, we restrict a ‘solid state’ of lattice bosons as a diagonal long-range ordered state breaking either the translational symmetry of the original lattice or the symmetry of different sublattices. The order is again of a striped or a SCDW type, to which we refer as striped order II. With quantum hopping present, SCDW phases have previously been found in models with next-nearest neighbor repulsion on the triangular lattice\(^{20}\) and plaquette interactions on the square lattice\(^{22}\). For hardcore bosons on the dual of the bowtie lattice, without the next-nearest neighbor repulsion or plaquette interactions, we shall show that striped phases also emerge.

At \( \mu/V = 2 \), a special degeneracy of the ground states appears, as illustrated in Fig. 2. Since breaking one of the two stripe ordered patterns along the \( y \)-direction costs energy, no interface can be formed along the horizontal direction at zero temperature. The lattice can be divided into blocks along the \( x \)-direction without creating any

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FIG. 1. (Color online). The dual of the bowtie lattice consisting of \( n_1 \times n_2 = 4 \times 6 \) unit cells with periodic boundary conditions. The unit cell is shown in the top, where A, B and C label the three sublattices. The A and B sublattices are related by symmetry, and \( a_1 \) and \( a_2 \) are primitive vectors with length 1 and 2/3 respectively. The numbers arranged in horizontal (vertical) direction are the \( x(y) \) coordinates of the lattice sites in units of 1/3. The linear size of the system is \( L_x = L_y = 4 \), with \( N = 72 \) sites. The coordination number of sites in the A and B sublattices is 3, and that of sites in the C sublattice is 4. (a) and (b) show the \( p = 1/2, 2/3 \) SCDW phases respectively, where black circles represent bosons and white ones stand for vacant sites.
such energy costing interfaces. The length of a half filled block in the $x$-direction is 2, that of the $2/3$ filled one is 1. For a system with linear size $L$ in the $x$-direction, which is restricted to be even in this work, let $n$ and $L - 2n$ be the number of half filled blocks and $2/3$ filled blocks respectively, the energy density (per site) of the system is $E = (-3n\mu + (L - 2n)(-2\mu + V))/(3L) = -V$, which is independent of $n$. Thus, there are $L/2 + 1$ possible boson densities (per site): $\rho_n = (2L - n)/(3L), n = 0, 1, \ldots, L/2$. For each filling, there is a total of $D(\rho) = L(L - n - 1)!(L - 2n)!/n!$ degenerate states. In the thermodynamic limit, the system can have any density between 1/2 and 2/3. Without quantum hopping $t$, since long-range order offers no energy advantage, all these states may exist with equal probability at the temperature closing to the limit of zero. The zero temperature entropy per site can be found as

$$s = \frac{k_B \ln \sum_{\rho} D(\rho)}{N}$$

$$= \frac{k_B \ln[2 \cosh(L \sinh^{-1}(0.5))] }{(9L^2/2)},$$

which goes to zero as the system size $L$ tends to infinity ($k_B$ is the Boltzmann constant), which is expected by considering the quasi 1-dimensional character of the states. In the language of the spin-1/2 XXZ model, the long-range magnetic order is destroyed by the competition between the staggered field and the exchange in the $z$ direction, instead of the geometrical frustration. This is different from the frustrated ground state on the triangular lattice at half filling.

III. QUANTUM PHASE DIAGRAM

Considering these interesting properties of the model introduced by the inhomogeneity and the anisotropy of the dual of the bowtie lattice, it is desirable to explore quantum properties of the model. To this end, we performed extensive simulations by using the stochastic series expansion (SSE) quantum Monte Carlo method with directed loop updates to the hardcore bosons on the dual of the bowtie lattice.

We start with the simple limit $V = 0$. The phase diagram is shown in Fig. 3. We find that the system is in a striped superfluid phase (SSF) with striped order II and nonzero superfluid stiffness between a Mott insulating phase (MI) and an empty phase, except for $\mu = 0$ where the model has an exact particle-hole symmetry and sits in a superfluid phase.

The phase diagram for $V \neq 0$ is much richer, see Fig. 4. We find SCDW I with $\rho = 1/2$ and SCDW II with $\rho = 2/3$ when the hopping is weak, as expected. The phase transition between the two SCDW phases is first order. A stable SSF phase emerges outside the two SCDW phases, except for $\mu = 0$.

We will now proceed to discuss the phase diagrams in more detail.

To characterize density wave order, one usually measures the static structure factor $S(Q)/N$ with $Q$ the wavevector

$$S(Q)/N = \frac{1}{N^2} \left\langle \sum_k n_k e^{iQ \cdot r_k} \right\rangle^2,$$

where $k$ labels sites, and $N$ is the total number of sites. However, in our case, it is more convenient to distinguish the two striped phases by introducing the following quasi-structure factors, which measure the density differences between different sublattices, or, in the language of the XXZ model, the various magnetic orders in the $z$ direction:

$$S_\alpha/N = \frac{1}{N^2} \left\langle \left(\sum_i \epsilon_\alpha(n_i)\right)^2 \right\rangle.$$

FIG. 3. (Color online). Phase diagram at the limit $V = 0$. The right inset shows density wave: particle density per column as a function of the $x$ coordinate of the column, at $\mu = 1, t = 0.5$. The density wave is absent at $\mu = 0$, as shown in the left inset.
For \( \alpha = 1 \), \( e_i^{(1)} = (-1)^{x(i)} \), where \( x(i) \) is the \( x \) coordinate of site \( i \). \( S_1/N \) measures the square of the density difference between the two sublattices consisting of sites with even or odd \( x \) coordinate, showing a density wave at wave vector \((\pi, 0) \) if not zero. For \( \alpha = 2 \), \( e_i^{(2)} = 1 \), if \( \text{mod} (x(i), 3) = 1 \); \( e_i^{(2)} = -1 \), if \( \text{mod} (x(i), 3) = 2 \); \( e_i^{(2)} = 0 \), otherwise. \( S_2/N \) is the square of the density difference between the A and B sublattices. We also define the third quasi-structure factor

\[
S_3 = \frac{1}{N} \langle \sum_i e_i^{(3)} n_i \rangle,
\]

with \( e_i^{(3)} = 0 \), if \( \text{mod} (x(i), 3) = 1 \); \( e_i^{(3)} = 1 \), if \( \text{mod} (x(i), 3) = 2 \); \( e_i^{(3)} = -1 \), otherwise. \( S_3 \) measures the difference of the boson densities on the B and C sublattices. In the present model, we find that \( S_2 \) is always zero, which means that the symmetry between the A and B sublattices is always kept, thus \( S_3 \) is also the density difference of the bosons on the A and C sublattices.

The superfluid stiffness is measured in terms of winding number fluctuations\(^{24}\),

\[
\rho_\alpha^x = \frac{< W_\alpha^2 >}{\beta t},
\]

where \( \alpha \) labels the \( x \) or \( y \)-direction, \( W_\alpha \) is the winding number in the \( \alpha \) direction, and \( \beta \) is the inverse temperature. Typically the superfluid density \( \rho_s \) is the average of the two stiffnesses. Considering the anisotropy between the \( x \) and \( y \)-directions, we shall distinguish the superfluid stiffness in the two directions.

We measure the particle density, the quasi-structure factors and the superfluid stiffnesses as functions of the chemical potential \( \mu \) and the hopping amplitude \( t \) (in units of the nearest-neighbor repulsion \( V \), if \( V \neq 0 \)). In the simulations, we set \( L_x = L_y = L \), which is restricted to be even. The total number of sites is \( N = 9L^2/2 \). The inverse temperature was chosen as \( \beta = 6L/t \) to make sure that the simulations access the ground state properties.

### A. Non-interacting quantum system \((V = 0)\)

We first consider the non-repulsive limit \( V = 0 \). The phase diagram is shown in Fig. 3, which is invariant under the interchange of particles with holes \( \rho \rightarrow 1 - \rho \) and the change of sign \( \mu \rightarrow -\mu \).

Fig. 5(a) shows the boson density \( \rho \) as a function of the chemical potential \( \mu \) with hopping \( t = 1 \). We see that the density varies continuously from \( 0 \) to \( 1 \) as \( \mu \) changes from -10/3 to 10/3. This can be well understood in the single particle picture with 10/3 the average coordination number. In this region, \( \rho_x^s \) and \( \rho_y^s \) become nonzero with \( S_3 > 0 \) for \( \mu > 0 \), or \( S_3 < 0 \) for \( \mu < 0 \), as shown in Fig. 5(b) and (c). This means the system displays both DLRO and ODLRO. To further confirm this, we sample the average particle density at the \( x \)-th column

\[
\rho(x) = \langle n_i(x) \rangle /
\]

in the \( x \)-th column. A density wave at wavevector \((2\pi, 0)\) is seen in the right inset of Fig. 3 for \( \mu/t = 2 \). For \( \mu/t = 0 \), no such wave appears, as shown in the left inset. Thus the system indeed has a striped order of type II.

This result can be understood in the following way: At \( \mu = 10t/3 \), holes can appear in the system. Each hole costs a potential energy \( \mu \), and gains kinetic energy \(-10t/3\) by hopping freely. The wave functions of the holes spread out over the entire lattice, but the probability of finding a hole in the C sublattice is 4/3 times of that in the B (or A) sublattice due to the coordination number difference. The striped density wave thus starts...
to develop. At $\mu = 0$, the difference disappears due to the exact particle-hole symmetry. However, as mentioned above, this striped order II does not break the translational symmetry of the original lattice, nor the symmetry between the A and B sublattices. Thus, we would call this phase bearing both ODLRO and DLRO a striped superfluid, instead of a supersolid phase (SSF). This SSF phase is a result of the interplay of the chemical potential and the lattice inhomogeneity and anisotropy.

This picture can also be understood in the language of the XXZ model. At $\mu = 10t/3$, the xy components of spins start to align, forming a ferromagnetic phase due to the competition between the in-plane coupling and the external magnetic field $h = \mu$. As the field $\mu$ decreases, the ferromagnetic order in the $xy$ plane gets stronger, and meanwhile the magnetization in the $z$ direction decreases. The inhomogeneity of the lattice structure makes the local magnetization of $z$ component in C sublattice weaker than that in the A and B sublattices. This results in a staggered long-range ferromagnetic order in the $z$ direction. When the external field $\mu$ reaches 0, the magnetization and the staggered order in the $z$ direction disappears completely.

Another interesting phenomenon in this model is that the superfluid stiffness along the $x$-direction (perpendicular to the stripes) becomes larger than that along the $y$-direction (parallel to the stripes) near half filling. In contrast, a larger superfluid stiffness along the stripes than transverse to the stripes was reported in various striped supersolid states on the square and triangular lattices$^{8,11,19–21}$, where the lattice itself is isotropic. Clearly, the anisotropy of the superfluid stiffness reflects the inequivalence of the $x$ and $y$-directions of the dual of the bowtie lattice, but it is not clear a priori that which of them should be larger.

### B. Interacting quantum system ($V \neq 0$)

With the nearest-neighbor repulsion $V$ present, we have a different phase diagram, without the symmetry when interchanging particles with holes. In the single-boson picture, the system becomes empty when $\mu \leq -10t/3$, which is in agreement with our simulations. Considering a single freely hopping hole on the lattice, one can show that the system sits in the $\rho = 1$ Mott insulating state when $\mu \geq 10(t + V)/3$. Our simulations confirm this for $t/V \gg 1$, where the single hole added is almost free with its wavefunction spreading over the whole lattice. However, the MI boundary is curved when the system is approaching the classical transition point $\mu/V = 4$, $t = 0$. For small $t/V$, holes are created on the C sublattice, which costs the same chemical potential $\mu$ but gains a potential energy $4V$, more than for those created on the A or B sublattices. These holes are forced to hop along $y$-direction. The kinetic energy gain of a single hole by the second order hopping processes is $-4t^2/V$. Otherwise, the kinetic energy gain $t$ cannot compensate the cost of the potential energy. This explains the stronger superfluid stiffness along the $y$-direction, see Fig. 6(d), and results in the curved boundary: $\mu = 4V + 4t^2/V$.

We now show results obtained by scanning the chemical potential at constant $t/V$.

In Figs. 6(a)-(d), we plot the particle density $\rho$, the quasi-structure factors $S_1/N, S_3,$ and the superfluid stiffnesses $\rho_\mu, \rho_z$ as functions of $\mu/V$ for $t/V = 0.3$ respectively.

Between $\mu/V = 2$ and 3.7, the $\rho = 2/3$ SCDW II phase is found with superfluid density $\rho_s = 0$, which means that the bosons are localized. The quasi-structure factor $S_3 \approx 0.28$, which is less than the value $1/3$ for the exact static SCDW II phase due to the presence of hopping $t$.

At $\mu/V = 3.7$, the density starts to grow continuously as $\mu$ increases, indicating a second order phase transition. $S_3, \rho_\mu$ and $\rho_z$ are all finite in the region $3.7 < \mu/V < 4.6$. The model displays both DLRO and ODLRO. Again, since the striped order II does not break the translational symmetry of the original lattice, the model turns out to be in a SSF phase, which can be understood in terms of doping bosons in the SCDW II state, or equivalently, doping holes in the Mott insulating state, which tend to hop along the $y$-direction as described in previous text.

A SCDW I phase is clearly observed in the region $0.68 < \mu/V < 2$, with $\rho = 1/2$, $S_1/N = 0.193$, $S_3 = 0.014$, and $\rho_\mu, \rho_z$ converging to 0. Here, the striped order I shown in Fig. 1(a) is slightly adapted by an additional order II. We show the density per row $\rho(x)$ in the inset of Fig. 4. The ordering at wavevector $(\pi, 0)$, which breaks the translational symmetry of the original lattice, is clearly seen. Thus the bosons here form a true symmetry-broken solid state.

Doping this solid (SCDW I) with holes leads to a phase

![FIG. 6. (Color online). The particle density $\rho$, quasi-structure factors $S_1/N, S_3,$ and superfluid stiffnesses $\rho_\mu, \rho_z$ as functions of $\mu/V$, at the cut $t/V = 0.3$.

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The image contains a graph with the following axes and labels:
- **X-axis:** $\mu/V$
- **Y-axis:**
  - $S_1/N$
  - $S_3$
  - $\rho_\mu$
  - $\rho_z$

The graph shows the evolution of these quantities as $\mu/V$ varies. The inset of Fig. 4 illustrates the density per row $\rho(x)$ for the striped order II.
separation, instead of supersolidity. A density jump caused by a first order phase transition is observed at $\mu/V = 0.68$. The mechanism is similar to what was found for the square lattice and the triangular lattice$^{9,10}$. Adding $3L/2$ holes onto the SCDW I solid decreases the density infinitesimally in the thermodynamic limit. Each hole costs a chemical potential $\mu$ and gains a kinetic energy which is quadratic in $t$ when empty at SSF phase, which finally ends when the lattice becomes metastable. This situation is illustrated by the interesting multi-loop hysteresis curves for a $L = 4$ system at $t/V = 0.2$, see Fig. 8 (b). During the simulations, we store the last configuration of a finished simulation as the initial configuration of the next simulation with a new value of the chemical potential$^{20,25}$. Starting from $\mu/V$ much less than 2 to ensure that the system stays in the ground state, i.e. $\rho = 1/2$ SCDW I, we increase $\mu/V$ and sample the particle density $\rho$. The system does not jump to the real ground state ($\rho = 2/3$) immediately when $\mu/V$ passes the transition point $\mu/V = 2$. This curve is labeled as ‘up1’ in Fig. 8 (b). Then, we start with $\mu/V > 2$ to ensure that the system stays in the 2/3 filling SCDW II state, then decrease $\mu/V$. The system may jump to the $\rho = 1/2$ state which is the ground state, closing a hysteretic loop (the curve ‘down1’), or jump to a metastable phase with 7/12 filling by chance (the curve ‘down2’), when $\mu/V$ is small enough. We can use the latter configuration as the initial configuration, and increase $\mu/V$ again. It is seen that the system stays in the metastable state until $\mu/V$ reaches a value much larger than 2, as shown by the curve ‘up2’. Curves ‘down2’ and ‘up2’ form another hysteresis loop.

This interesting phenomenon makes the simulations near $\mu/V = 2$ very difficult. The data near this point, shown in Fig. 6, are obtained by initializing configurations in the ‘right’ way. The transition points between

![FIG. 7. (Color online). The $\rho = 1/2$ solid doped with holes. (a) Holes (green circles with dashed boundary) added in the solid. (b) Lining the holes costs no additional energy. (c) A domain wall (dashed red line) is introduced at no cost by shifting the right half of the lattice. (d) The holes can hop freely across the domain wall and gain kinetic energy.](image)

![FIG. 8. (Color online). Energy level crossing (a) and multi-loop hysteresis (b) for hardcore bosons on a $L = 4$ lattice at $\mu/V = 0.2$. Three phases with different densities coexist near $\mu/V = 2$.](image)
and B sublattices of the original lattice is broken in this translational symmetry nor the symmetry between the A and B sublattices of the original lattice is broken in this translational symmetry nor the symmetry between the A and B sublattices of the original lattice is broken in this translational symmetry nor the symmetry between the A and B sublattices of the original lattice is broken in this translational symmetry nor the symmetry between the A and B sublattices of the original lattice is broken in this translational symmetry nor the symmetry between the A and B sublattices of the original lattice is broken in this translational symmetry nor the symmetry between the A and B sublattices of the original lattice is broken in this translational symmetry nor the symmetry between the A and B sublattices of the original lattice is broken in this translational symmetry nor the symmetry between the A and B sublattices of the original lattice is broken in this translational symmetry nor the symmetry between the A and B sublattices of the original lattice is broken in this

The emergence of such a state is due to the inhomogeneity and the anisotropy of the lattice structure. However, neither the translational symmetry nor the symmetry between the A and B sublattices of the original lattice is broken in this

the SCDW I and SCDW II phases can be found from the energy level crossings.

Increasing hopping $t/V$ larger than 0.37, we find that the SSF phase emerges in the region between the two SCDW phases. Fig. 9 shows the case for $t/V = 0.4$. It is also clear that the transition from SCDW I to the SSF phase is of first order, and the transition from SCDW II to the SSF phase is continuous. In this region, the superfluid stiffnesses in the two directions become different with $\rho^x_s > \rho^y_s$ near half filling. This is also the case for the SSF state with $\rho < 1/2$ near half filling. For the SSF state with $2/3 < \rho < 1$, we see the same anisotropic behavior as that at the cut $t/V = 0.3$.

At $t/V = 0.5$ shown in Fig. 10, the 2/3 filling SCDW II phase is the only density wave phase. Although the finite size data of $S_1/N$ are nonzero in a large region, a finite-size scaling analysis indicates that they finally converge to zero as system size turns to infinity, as illustrated in Fig. 11. The strong hopping destroys the SCDW I order. The SSF state is found outside the SCDW II phase for $\mu/V < 2.65$ and $\mu/V > 3.5$. We still see a large anisotropy of superfluid stiffness in the SSF states: $\rho^x_s > \rho^y_s$ near half filling, and $\rho^x_s > \rho^y_s$ for $2/3 < \rho < 1$.

**IV. CONCLUSION**

We have investigated the ground state behavior of hardcore bosons on the dual of the bowtie lattice. A special state (SSF) with a striped order II and an ODLRO is found even in the absence of repulsion. The emergence of such a state is due to the inhomogeneity and the anisotropy of the lattice structure. However, neither the translational symmetry nor the symmetry between the A and B sublattices of the original lattice is broken in this

state. Such SSF states should exist on anisotropic and inhomogeneous lattices with sites having different coordination numbers. Including a nearest-neighbor repulsion causes a much richer phase diagram. Two SCDW phases with different striped order are found. A SSF phase exists outside the two SCDW phases, between the Mott insulating phase and the empty phase. Comparing with the SSF state with $V = 0$, we see that the striped order II is enhanced greatly by the repulsion in the SSF state. The phase transition between the two SCDW phases is first order. The transition between the SCDW I and SSF state is also first order, while the transition between SCDW II and SSF state is continuous. We also report anisotropies of superfluid stiffness in the SSF states with and without repulsion. In the SSF state with repulsion,
the superfluid stiffness shows different anisotropies in the region near half filling and above 2/3-filling. In conclusion, hardcore bosons on inhomogeneous lattice present rich and interesting properties. With the help of the new developed hologram projecting technique, it is now possible to generate complex optical lattices such as the present dual of bowtie lattice. It should therefore be worthwhile and practicable to detect the SSF state and verify the predicted properties of the model experimentally.

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