Tricritical $O(n)$ Models in Two Dimensions

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1 O(\(n\)) model
   - O(\(n\)) model and its loop representation
   - Exact results of O(\(n\)) loop model on honeycomb lattice
   - Exact results of O(\(n\)) loop model on square lattice

2 Tricritical O(\(n\)) loop model
   - An O(\(n\)) loop model with vacancies on honeycomb lattice
   - Exact characterisation of O(\(n\)) tricriticality in 2D
   - Numerical verifications
   - Mapping Low-T O(\(n\)) model onto tricritical O(\(n\)) model
   - Numerical Verifications

3 Summary
1. **O(n) model**
   - O(n) model and its loop representation
   - Exact results of O(n) loop model on honeycomb lattice
   - Exact results of O(n) loop model on square lattice

2. **Tricritical O(n) loop model**
   - An O(n) loop model with vacancies on honeycomb lattice
   - Exact characterisation of O(n) tricriticality in 2D
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   - Numerical Verifications

3. **Summary**
O(n) Spin Model

Boltzmann weight

\[ e^{-\beta H} = \prod_{\langle i,j \rangle} (1 + x \vec{S}_i \cdot \vec{S}_j) \]

\(\vec{S}_i\): n-component vector spin

The weight is O(n) symmetric

n = 1, the Ising model
n = 2, the XY model
n = 3, the Heisenberg model
O(n) Loop Model

Partition function of the O(n) spin model:

\[ Z_{\text{spin}} = \int \prod_{<i,j>} (1 + x\vec{S}_i \cdot \vec{S}_j) \prod_k d\vec{S}_k \]

Expansion of \( Z \) in powers of \( x \) yields an O(n) loop model

\[ Z_{\text{loop}} = \sum_G x^{N_b} n^{N_l} \]

\( G \): configuration of loops, \( N_b \): the number of bonds (or total number of vertices visited by loops), \( N_l \): the number of loops
A simple configuration of $O(n)$ loop model

Weight: $x^{16}n^2$

Now $n$ can be a real number

$n = 0$, the Self-avoiding Walk or the Polymer
phase diagram of $O(n)$ loop model on honeycomb lattice

On honeycomb lattice, this model is solvable at

$$x = \frac{1}{\sqrt{2 \pm \sqrt{2 - n}}}$$

$+$: critical line, $-$: describes the low-temperature phase
Universal parameters known.
On square lattice

The same universal behavior was also found and solved for square-lattice $O(n)$ model.

- A critical branch: transition between dense phase and dilute phase
- A Low temperature branch: low temperature phase
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3 Summary
Central questions of this talk

- Does the $O(n)$ model permit a tricritical point?
- Can we calculate its exponents as function of $n$?
Vacancies

Spin-1 Ising model: \( s_i \in -1, 0, 1 \)

\[
\mathcal{H} = - \sum_{\langle i,j \rangle} J s_i s_j + \sum_k \mu s_k^2
\]

- For \( \mu \) small, a continuous transition.
- Larger \( \mu \) leads to a discontinuous transition.
- The two regimes are separated by a tricritical point

\( s_i = 0 \) can be considered as a vacancy, the model is an \( O(1) \) model with vacancies.

Vacancies can lead to tricriticality
New model: O(n) loop model with vacancies

For general $n$, we introduce $v$ the probability that a face vacant

$$Z_{spin} = \sum_{\mathcal{L}} v^{N_v} (1-v)^{N-N_v} \int \prod_{i|\mathcal{L}} d\vec{S}_i \prod_{\langle ij \rangle} (1 + x \vec{S}_i \cdot \vec{S}_j)$$

Mapping onto the loop representation:

$$Z_{loop} = \sum_{\mathcal{G}} v^{N_v} (1-v)^{N-N_v} x^{N_b} n^{N_l}$$

$N_v$ the number of vacant faces, $N$ the number of total faces
A configuration of the loop model with vacancies

weight: $v^2(1 - v)^{18}x^{16}n^2$
Expected schematic phase diagram

At small $v$ likes 'old' $O(n)$ model

At large $v$ and large $x$, 'Vacant' lattice phase and lattice full with loops coexist.

We expect:

a tricritical point separating the critical line and first order transition line
Postulation

We can not solve this model analytically, but there are clues:

- Fully packed $O(n)$ loop model $\iff$ Critical $q = n^2$-state Potts model

- Critical $O(n)$ model $\iff$ Tricritical $q = n^2$-state Potts model

- Postulation:
  Tricritical $O(n)$ model $\iff$ Tri-tricritical $q = n^2$-state Potts model (?)
As the strength of 4-spin interaction increasing, Tricritical point itself becomes discontinuous.

Exactly solved at Tri-tricritical point, universal parameters are known

Nienhuis, Warnaar and Blöte, 1992
Exact properties proposed

Replacing $\sqrt{q}$ with $n$ in the solutions of tri-tricritical $q$-state Potts model, we have

- **Conformal anomaly:**
  \[
  \Delta - \frac{1}{\Delta} = n \implies 2 \cos \frac{\pi}{m+1} = \Delta \implies c = 1 - \frac{6}{m(m+1)}
  \]

- **Scaling dimensions $X_j$**
  \[
  \Delta \implies \Delta_j = 2 \cos \frac{k_j \pi}{m+1} \implies X_j = \frac{k_j^2 - 1}{2m(m+1)},
  \]

where $\Delta_1 = 1/\Delta$, $\Delta_2 = -1/\Delta$

$X_1$ is identified as $X_m$, and $X_2$ as $X_t$ later numerically.
$X_h$ is given by the entry $(m/2, m/2)$ in the Kac formula
Numerical verifications

Now we build transfer matrix for our loop model with vacancies

1. determining tricritical points by solving finite-size scaling equations

\[ X_i(v, x, L) = X_i(v, x, L - 1) = X_i(v, x, L - 2) \]

\[ X_i(L): X_h(L), X_t(L) \text{ or } X_m(L). \]

The solutions converge to the tricritical points.

2. determining scaling dimensions and conformal anomaly at the tricritical points:

\[ f(L) = f(\infty) + \pi c/(6L^2) \]

\[ X_i(v(L), x(L), L) = X_i + cuL^{y_u} + \cdots \]
Comparing with theoretical prediction

\[ c(\text{num}: +; \text{theor}: \text{dashed line}) \]
\[ X_t(\text{num}: \times; \text{theor}: \text{dotted line}) \]
Comparing with theoretical prediction II

\[ X_h(\text{num} : \times; \text{theor} : \text{dotted line}) \]
\[ X_m(\text{num} : +; \text{theor} : \text{dashed line}) \]

see: Guo, Nienhuis, Blöte, PRL 96, 2006
However, this model fails to generate tricriticality on square lattice. In addition, we wish to find an exact basis for the characterisation. This is doable:

the exactly solved Low-T branch of O(n) model can be mapped onto a tricritical O(n’) model with different loop weight \( n’ = n - 1/n \).
Transformations:

The steps:

- Loop model with loop weight $n$
- $\downarrow$
- Discrete state model (ADE model)
- $\downarrow$
- Another loop model with vacancies and with different loop weight $n'$ (a tricritical $O(n')$ model)
Pasquier’s trick: loop model $\rightarrow$ ADE model

Making Use of

Adjacency diagram: e.g.:

```
 a   b   c
```

Adjacency matrix: $A_{ij} = 1$ if $i$ and $j$ are adjacent; 0 otherwise

$$
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
$$

$\Lambda = \sqrt{2}, \quad (S_1, S_2, S_3) = (1, \sqrt{2}, 1)$

Define a 'spin' model:

Spins on either side of the loop take states that are neighbors on the adjacency diagram.
The Boltzmann weight of an ADE configuration is defined in terms of local vertex weights of the loop model, multiplied with a product on every bend of the loops:

\[ W_{ADE} = x^{N_b} \prod_{\text{bend}} A_{ij} \left( \frac{S_j}{S_i} \right)^\gamma_{\text{bend}}/2\pi \]

state \( j \) sits inside the loop, \( i \) outside. The sum of \( \gamma \) along a loop is \( 2\pi \)
The partition function of the ADE model is

\[ Z_{ADE} = \sum_G \sum_{S|G} W_{ADE} = Z_{\text{loop}} \]

where the first sum is on all loop configurations and the second one on the allowed choices for the ADE states.

- Sum first on the states of the innermost domains. This results in \( \Lambda \).
- The domains summed over can now be ignored and we sum over the domains that now remain without enclosures.
- Finally, we get Boltzmann weight of a loop configuration, with loop weight \( \Lambda \).
A simple example to show the equivalence

\[ W_{ADE} = x^6 A_{12} \frac{S_2}{S_1} = x^6 \sqrt{2} \]

\[ W_{ADE} = x^6 A_{21} \frac{S_1}{S_2} + x^6 A_{23} \frac{S_3}{S_2} = x^6 \sqrt{2} \]

\[ W_{ADE} = x^6 A_{32} \frac{S_2}{S_3} = x^6 \sqrt{2} \]

Sum on out-most 'spins' gives a trivial constant factor to the weight of the loop configuration.
First: $O(n)$ loop model $\mapsto$ ADE model using adjacency diagram $E_6$

We chose $E_6$ (a chicken foot diagram):

![Diagram](image.png)

The number $q$ of branches is chosen $q = (n - 1/n)^2$

Such that the largest eigenvalue of $A$ is $n$. 
Adjacency matrix $\tilde{A}$

$A$ reduces to:

$$
\tilde{A} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & q & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
$$

Largest eigenvalue $\Lambda = n$

associated eigenvector: $(S_a, S_b, S_c, S_d) = (1, n, \frac{n}{n-1/n}, \frac{1}{n-1/n})$
Second: ADE model $\leftrightarrow$ loop model with vacancies

we interpret the extrema of $E_6$, i.e. $a$ and $d$ as vacancies.

\[
x \cdot N_b \prod_{bend} A_{i,j} \left( \frac{S_j}{S_i} \right)^{\gamma_{bend}/2\pi} = x \cdot N_x \cdot v_1^{N_{v_1}} \cdot v_2^{N_{v_2}} \prod_{bend'} A_{i',j'} \left( \frac{S_{j'}}{S_{i'}} \right)^{\gamma_{bend'}/2\pi}
\]

$i'$, $j'$ are $b$ or $c$ types.
Loop weight of the new loop model

Given a loop configuration, sum on allowed ADE states

$$\sum_{S|G} x^{N_b} \prod_{\text{bend}} A_{ij}(\frac{S_j}{S_i})^{\gamma_{\text{bend}}/2\pi} = x^{N_b} n^{N_l}$$

Given a loop configuration with vacancies, sum on allowed ADE states, some domains have to be $a$ or $d$ type, others $b$ or $c$ type,

$$\sum_{S|G'} x^{N_x} v_1^{N_{v_1}} v_2^{N_{v_2}} \prod_{\text{bend'}} A_{i'j'}(\frac{S_{j'}}{S_{i'}})^{\gamma_{\text{bend}'}/2\pi} = x^{N_x} v_1^{N_{v_1}} v_2^{N_{v_2}} \times (?)^{N_l}$$

What is the loop weight of this new loop model with vacancies?
Derivation of the new loop weight I

postulation: The probability that a domain is in state $k$ equals

$$P(k) = \frac{S_k^2}{\sum_j S_j^2}$$

We can prove that
If the postulation holds for the state in the innermost domain, it also holds for the state in the next-to-innermost domain and vice versa. This consistency argument can be repeated recursively. Thus, the postulation is correct.
Derivation of the new loop weight II

Now consider:

The probability $Q(j)$ that a domain is in state $j$, conditional on it being non-vacant states $b$ or type $c$, is

$$Q(j) = \frac{S_j^2}{\sum_k S_k^2} \frac{S_j^2 + qS_c^2}{\sum_k S_k^2} = \frac{S_j^2}{S_b^2 + qS_c^2}$$

Assuming the new loop weight is $n'$, the probability $Q(j, k)$ that $j$ surrounds state $k$, conditional on both $j$ and $k$ being non-vacant, is

$$Q(j, k) = Q(j) A_{jk} \frac{S_k}{S_j n'}$$
Derivation of the new loop weight II

Using the normalization condition:

\[ \sum_{j=b,c} \sum_{k=b,c} Q(j, k) = 1 \]

We find that \( n' = n - 1/n \)
Universal properties

Conformal anomaly for Low-T $O(n)$ model is known as

$$n = 2 \cos\left(\frac{\pi}{m + 1}\right), \quad m \geq 1, \quad c = 1 - \frac{6}{m(m + 1)}$$

Note $n' = n - 1/n$, compare with the postulation for tricriticality

$$2 \cos\left(\frac{\pi}{m + 1}\right) = \Delta, \quad c = 1 - \frac{6}{m(m + 1)}$$

where $n' = \Delta - 1/\Delta$

We find a Tricritical $O(n')$ model with exactly known tricritical points as function of $n'$
Why chicken foot diagram?

Requirement:
All non-vacant states are adjacent to vacant states

Such that Vacancies can sit in everywhere

A counter example:
Tricritical loop model on square lattice

Following the same steps, we have tricritical loop model on square lattice

\[ W_1 = 1, \quad W_2 = u, \quad W_3 = v, \quad W_4 = w, \quad W_5 = wn^{-1/4}, \]
\[ W_6 = un^{-1/4}, \quad W_7 = v, \quad W_8 = w(n^{1/2} + n^{-1/2}), \]
\[ W_9 = un^{1/4}, \quad W_{10} = 1 \]

Where \( u, v, w \) are known weights of Low Temperature branch.
Numerical Verifications

We build transfer Matrix for the new loop model both on honeycomb lattice and on square lattice.

Conformal anomaly and scaling dimensions are calculated at the suggested tricritical locus. The final estimates agree convincingly with the theoretical predictions.

Data for tricritical $O(1/2)$ model on square lattice vs. theoretical values

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outlines

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Summary

- Exact tricritical $O(n)$ universal parameters in two dimensions as a function of $n$ were found. They apply to honeycomb lattice and square lattice, and are believed universal.

- The exactly solved Low-temperature branch of the 2D $O(n)$ model can be mapped onto a tricritical $O(n)$ model with a different value of $n$.

- The exact tricritical points of this model were found both on honeycomb lattice and square lattice.

- Vavancy-vacancy couplings are important on square lattice. No such coupling in the original model with vacancies leads to the absence of tricriticality in the square lattice model.
Thank You