O($n$) loop models and recent developments on discrete holomorphy

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Outline

This talk has three main ingredients:

1. Exactly solved (integrable) $O(n)$ loop models in statistical mechanics.

2. Random self-avoiding walks (SAW) on a planar lattice.

3. Discrete holomorphicity of a parafermionic observable.

There have been some remarkable developments relating these topics.
Interactions with Henk Blöte

Leiden 1987
Self-avoiding walks

The study of lattice SAW has a long history, see, e.g., the book by Madras & Slade.

A SAW is a random walk on a lattice in which no vertex is visited more than once (self-avoiding).

The SAW is a good and long studied model of a polymer chain. It poses a number of challenging questions in mathematics with regard to

1. combinatorics,
2. geometry.
Combinatorics of SAW

Let $C_N$ be the number of $N$-step SAW of length $N$.

$C_N$ has been enumerated for all imaginable lattices. For example, on the square lattice

$$C_{71} = 4190893020903935054619120005916$$


It is expected that

$$C_N \sim \mu^N N^{\gamma - 1}$$

where

- $\mu$ is the connective constant (lattice dependent),
- $\gamma$ is a configurational exponent (universal).
Nienhuis (1982) considered the $O(n)$ model on the honeycomb lattice and obtained the exact values

$$\mu = \sqrt{2 + \sqrt{2}} \quad \text{and} \quad \gamma = \frac{43}{32}$$

at $n = 0$ via mappings between various models for the Boltzmann weights and Coulomb gas arguments for the exponents.
Bethe Ansatz integrable $O(n)$ models


The $O(n)$ model on the honeycomb lattice was solved exactly by means of the Bethe Ansatz.

\[ Z = \sum_{G} x^L n^P \]

where the sum is over all configurations $G$ of loops, with $L$ the length of the walk and $P$ the number of closed loops.

Exactly solvable for $x = x_c(n)$ where

\[ 1/x_c(n) = \sqrt{2 + \sqrt{2 - n}}, \quad n \in [-2, 2]. \]
This model is a special case of a more general model on the square lattice.

The underlying model is the 19-vertex model found by Izergin and Korepin (1983) – a.k.a. the $A_2^{(2)}$ vertex model in the Lie-algebraic classification of integrable models.

The more general loop model is the dilute O($n$) model on the square lattice, which reduces to the O($n$) model on the honeycomb lattice in two special limits of the spectral parameter.

We will return to this soon.

The 3-12 and martini lattices

The $O(n)$ model on the honeycomb lattice can be mapped to the $O(n)$ model on the semi-regular (b) 3-12 and (c) martini lattices.

Exact results follow for SAWs. For the 3-12 lattice, the exact value $\mu = 1.711\ 041\ldots$ follows from the solution of

$$\frac{1}{\mu^2} + \frac{1}{\mu^3} = \frac{1}{\sqrt{2 + \sqrt{2}}}$$

And for the martini lattice, the exact value $\mu = 1.750\ 564\ldots$ follows from the solution of

$$\frac{1}{\mu^3} + \frac{1}{\mu^4} = \frac{1}{\sqrt{2 + \sqrt{2}}}$$

A major breakthrough occurred with the connection between the scaling limit of SAW and Schramm-Loewner evolution (SLE).

**Theorem (loosely stated).** If the scaling limit of the 2-dimensional SAW exists and has a certain conformal invariance property, then the scaling limit must be $\text{SLE}_{8/3}$.

This theorem identifies the stochastic process $\text{SLE}_{8/3}$ as the candidate scaling limit.

Known properties of $\text{SLE}_{8/3}$ lead to calculations that rederive the value $\gamma = \frac{43}{32}$, assuming that $\text{SLE}_{8/3}$ is indeed the scaling limit.

Numerical verifications that $\text{SLE}_{8/3}$ is the scaling limit have been performed. However, the theorem makes a conditional statement, and the existence of the scaling limit (and therefore also a proof of its conformal invariance) remains an open problem.
In a surprising development Duminil-Copin and Smirnov came up with a rigorous proof of the Nienhuis result for $\mu$!

**Theorem** The connective constant for SAW on the honeycomb lattice is $\mu = \sqrt{2 + \sqrt{2}}$.

The proof consists of several parts:

1. Establishing a local identity for a parafermionic observable – the condition of discrete holomorphicity – from which the Boltzmann weight $x_c = 1/\mu$ appears naturally.
2. Establishing a global identity linking SAW generating functions in a domain of the lattice.
3. Proving that $\mu = \sqrt{2 + \sqrt{2}}$ amounts to showing that $Z(x) = +\infty$ for $x < 1/\mu$ and $Z(x) < +\infty$ for $x > 1/\mu$.
4. The latter builds on work by Hammersley and Welsh (1962) on excursions.
Smirnov also extended the local identity to the general $O(n)$ model on the honeycomb lattice with $n \in [-2, 2]$.

This gave an alternative way of obtaining the values of the critical point

$$\frac{1}{x_c(n)} = \sqrt{2 + \sqrt{2 - n}}.$$
SAW at a boundary


The $O(n)$ model on the honeycomb lattice with a boundary is also solved exactly by means of the Bethe Ansatz.

The integrable boundary weights follow from the solution of the reflection or boundary Yang-Baxter equation.

\[ Z = \sum_G x^L y^{L_s} n^P \]

where the sum now includes walks on the boundary of length $L_s$ with surface fugacity $y$.

Exactly solvable for $x = x_c(n)$ and $y = y_c(n)$ where

\[ y_c(n) = 1 + \frac{2}{\sqrt{2 - n}}. \]
Polymer adsorption transition

The model undergoes a special surface transition at \( y = y_c(n) \).

In particular, at \( n = 0 \) the value \( y_c = 1 + \sqrt{2} \) is the polymer adsorption transition.

The SAW with step fugacity \( x_c = 1/\sqrt{2 + \sqrt{2}} \) is

- adsorbed if \( y > 1 + \sqrt{2} \)
- desorbed if \( y < 1 + \sqrt{2} \).

A wealth of configurational exponents are also known.
Rigorous proof


**Theorem** The critical surface fugacity for SAW on the honeycomb lattice is \( y_c = 1 + \sqrt{2} \).

The rigorous proof given by Beaton *et al.* extends the argument used by Duminil-Copin and Smirnov to include a boundary.

This gives an alternative way to obtain \( y_c(n) \).

And most importantly, the second part of the paper provides the machinery necessary to rigorously prove that \( y_c \) is indeed the critical surface fugacity at \( n = 0 \).
The other orientation

There is another orientation of the honeycomb lattice:

\[ \exp \left( \frac{\mathcal{E}}{kT_a} \right) = 1 + \sqrt{2} = 2.414... \]

\[ \exp \left( \frac{\mathcal{E}}{kT_a} \right) = \sqrt[4]{\frac{2 + \sqrt{2}}{1 + \sqrt{2} - \sqrt{2 + \sqrt{2}}}} = 2.455... \]

rigorous proof by N.R. Beaton, arXiv:1210.0274
Discrete holomorphicity and integrability

Without doubt, the remarkable recent progress on establishing rigorous proofs for $\mu$ and $y_c$ for SAW on the honeycomb lattice is related to the integrability of the underlying lattice model.

In a series of papers, Cardy and his collaborators uncovered a remarkable connection between discrete holomorphicity and integrability for a number of well known models.

More precisely, they set up a parafermionic “observable” on the lattice which obeys the discrete Cauchy-Riemann equations. This condition leads to a system of linear equations which can be solved to yield the Boltzmann weights of the underlying lattice model.

Now these are the well known Boltzmann weights which satisfy the star-triangle or Yang-Baxter equations at criticality!

As an example, let’s look at the dilute $O(n)$ model on the square lattice.
The $O(n)$ loop model

\[ Z_{\text{loop}} = \sum_G \prod_{i=1}^9 m_i \rho_i^{m_i} n^P \]

Sum over all loop configs $G$

$m_i := \# \text{ occurrences of weight } \rho_i$

$P := \text{ total no of closed loops of fugacity } n$
$Nienhuis$ (1989)

\[
\rho_1 = \sin(3\lambda - u) \sin u + \sin 2\lambda \sin 3\lambda \\
\rho_2 = \rho_3 = \varepsilon_1 \sin(3\lambda - u) \sin 2\lambda \\
\rho_4 = \rho_5 = \varepsilon_2 \sin u \sin 2\lambda \\
\rho_6 = \rho_7 = \sin(3\lambda - u) \sin u \\
\rho_8 = \sin(3\lambda - u) \sin(2\lambda - u) \\
\rho_9 = -\sin(\lambda - u) \sin u
\]

with $\varepsilon_1^2 = \varepsilon_2^2 = 1$, $n = -2 \cos 4\lambda$
and $0 < u < 3\lambda$.

\[ F(z) = \sum_{G \in \Gamma(0,z)} P(G) e^{-i\sigma W(G)} \]

- \( P(G) \) is the probability of configuration \( G \)
- \( \Gamma(0,z) \) is the set of loop configurations for which points 0 & \( z \) belong to the same loop
- \( W(G) \) is the winding angle of the loop from 0 to \( z \)
- \( \sigma \) is the spin of the parafermion \( F(z) \)

\[ F(z) = \sum_{\gamma(a\rightarrow z) \subset \Omega} e^{-i\sigma W(\gamma(a\rightarrow z))} \rho_1^{m_1} \cdots \rho_g^{m_g} n^p \]

For some SAW \( \gamma(a\rightarrow z) \) across some domain \( \Omega \)
The parafermion is discretely holomorphic, i.e.

\[ F(p) - e^{i\theta} F(q) - F(r) + e^{i\theta} F(s) = 0 \]

on each vertex

\[ \begin{array}{c}
    d \\
    \theta
\end{array} \quad \begin{array}{c}
    r \\
    s \\
    b
\end{array} \]

Equivalent to requiring that \( F \) is divergence and curl free.
To establish this, choose a particular mid edge \( p \)
There are 4 possible different external connectivities.
Results in 4 eqns

\[ \rho_1 + \nu \rho_2 - \nu \xi^{-1} \rho_4 - \rho_7 = 0 \]
\[ -\xi^{-1} \rho_2 + \eta \rho_5 + \eta \nu \rho_7 - \nu \xi^{-1} (\rho_8 + \eta \rho_9) = 0 \]
\[ \eta \rho_3 - \xi \rho_4 - \nu \xi^{-2} \rho_7 + \nu (\eta \rho_8 + \rho_9) = 0 \]
\[ -\nu \xi^{-2} \rho_2 + \nu \xi \rho_4 + \eta \rho_6 - \xi^{-2} \rho_8 - \xi^{-2} \rho_9 = 0 \]

Here \( \nu = e^{i \theta (\beta + 1)} \) & \( \xi = e^{i \sigma \pi} \)

These \textbf{linear} eqns are satisfied by the bulk weights provided:

\[ \sigma = 1 - \frac{3 \lambda}{\pi} \]
\[ \eta = \sigma (\theta - \pi) + \theta \]

In particular, at \( \theta = \frac{\pi}{2} \)
\[ \eta = 3 \lambda / 2 \], the isotropic point.
A simpler world?

The surprising point is that the Yang-Baxter equations are \textit{cubic} in the Boltzmann weights, whereas the discrete Cauchy-Riemann equations are \textit{linear} in the Boltzmann weights.

There thus appears to be a simpler route to uncovering the integrable lattice models at criticality.

Why is this?

Fendley has pointed out that topology is a key ingredient in establishing the precise link between discrete holomorphicity and integrability.

We recently looked at this connection more closely in the context of the $\mathbb{Z}_N$ models, which can be discussed purely from the algebraic point of view.

The $\mathbb{Z}_N$ models


The $\mathbb{Z}_N$ model is an $N$-state model in which the spins $s_r$ at vertex $r$ take values from the $N$th roots of unity, $s_r = \omega^{q_r}$, with $q_r \in \{0, 1, \ldots, N - 1\}$ and $\omega = \exp(2\pi i/N)$.

The model contains a number of well known special cases:

- the Ising model ($N = 2$),
- the three-state Potts model ($N = 3$),
- the Ashkin-Teller model ($N = 4$).

The $\mathbb{Z}_N$ model has a well known set of critical points obtained by Fateev and Zamolodchikov (FZ) as solutions to the star-triangle relations.

By defining lattice parafermions as products of neighbouring order and disorder variables with a suitable phase factor, it was shown that the FZ Boltzmann weights ensure that the lattice parafermions obey the discrete version of the C-R equations.
The underlying motivation for this work on the $Z_N$ model is the construction of discretely holomorphic parafermions, which are expected to become the holomorphic parafermions of the CFT.

Our interest is in the connection with integrability. From that perspective, the key finding is that the contour sum around each elementary face or rhombi of the lattice vanishes, the key ingredient for which is the set of FZ critical Boltzmann weights.

This approach was extended by explicitly considering the condition of discrete holomorphicity on two and three adjacent rhombi:

- For two rhombi this leads to a quadratic equation for the Boltzmann weights which implies the known inversion relations for the $Z_N$ model.
- For three rhombi it leads to a cubic equation from which the star-triangle relation follows.

The simplicity of the discrete holomorphic approach is that the two- and the three-rhombus equations are equivalent to the one-rhombus equation.
Other recent work

- Extension to open boundary conditions compatible with discrete holomorphicity for the $Z_N$, dilute $O(n)$ model and the $C_2^{(1)}$ loop model.

Some new sets of boundary weights have been obtained.


- Identification of non-local currents with discrete holomorphic observables in the context of quantised affine algebras.

The bulk discrete holomorphicity relation and its boundary analogue are shown to be equivalent to conservation laws for non-local currents.

The “linearisation” of the YBE is explained in terms of the universal $R$-matrix of a quantised affine algebra, which by definition satisfies a linear relation.