Renormalization of Tensor Network States

Partial order and finite temperature phase transition in the Potts model on irregular lattice

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Background: challenge in the study of quantum many-body systems

- **No obvious small parameters: non-perturbative**
  Conventional methods of quantum field theory are not applicable

- **The total degree of freedom grows exponentially with system size**
  Numerical study meets the “Exponential Wall”

How to surpass the exponential wall?

\[ 1 + 2 + 2^2 + 2^3 + \ldots + 2^{100} + \]
Weak coupling approach

Convert a many-body problem into a one-body problem

✓ Mean field approximation

✓ Density Functional Theory
  • Most successful numerical method for treating weak coupling systems
  • Based on LDA or other approximations, less accurate
Strong coupling approach

Use a finite set of many-body basis states to treat a correlated system

- **Quantum Monte Carlo**
  - Suffer from the “minus-sign” problem

- **Numerical renormalization group**
Numerical renormalization group

Stage I: Wilson NRG 1975 -

0 Dimensional problems
(single impurity Kondo model)

Stage II: DMRG 1992 -

Quasi-1D quantum lattice models

Stage III: Tensor Renormalization group

Renormalization of tensor network states/models

2D quantum lattice model
2D or 3D classical statistical models
What is a tensor network state?
The partition function of a lattice statistical model with local interactions can be represented as a tensor–product model.

\[ Z = \text{Tr} \prod_i T_{x_i x'_i y_i y'_i} \]
Example: Tensor-network representation of the Ising model

\[ H = -J \sum_{\langle ij \rangle} S_i S_j \]
Quantum Lattice Model

d-dimensional quantum model = (d+1)-dimensional classical model under the framework of path integration

Like a classical model
Solve a tensor network model by the renormalization group method

Z. Y. Xie et al, PRL 103, 160601 (2009)
Z. Y. Xie et al, PRB 86, 045139 (2012)
Step 1: coarse graining

To contract two tensors into one

\[ M^{(n)}_{xx'yy'} = \sum_i T^{(n)}_{x_1'x_1yi} T^{(n)}_{x_2'x_2iy'} \]

\[ x = (x_1, x_2), \quad x' = (x'_1, x'_2) \]
Coarse graining tensor renormalization

Step 2: determine the unitary transformation matrices

By the higher order singular value decomposition

\[ M^{(n)}_{xx',yy'} = \sum_{ijkl} S_{ijkl} U^L_{xi} U^R_{x'j} U^U_{yk} U^D_{y'l} \]

Higher order singular value decomposition
Coarse graining tensor renormalization

Step 3: renormalize the tensor

cut the tensor dimension according to the norm of the core tensor

\[
T_{xx'yy'}^{(n+1)} = \sum_{ij} U_{ix}^{(n)} M_{ijyy'}^{(n)} U_{jx'}^{(n)}
\]
How accurate is the method?

Ising model on the square lattice  $D = 10$

Relative error of free energy

Temperature
Second renormalization of tensor network states

\[ Z = \text{Tr}(M M^{\text{env}}) \]

- **TRG:** truncation error of \( M \) is minimized
  
  But, what really needs to be minimized is the error of \( Z \)!

- **SRG:**
  
  minimize the error of the partition function
  
  The renormalization effect of \( M_{\text{env}} \) to \( M \) is included
How accurate is the method?

Ising model on the square lattice \( D = 10 \)

Relative error of free energy

- HOTRG
- HOSRG-Old
- HOSRG-New

Temperature

\( T \)
3D Ising model: magnetization

\[ M \sim t^\beta \]

HOTRG(D=14): 0.3295  
Monte Carlo: 0.3262  
Series Expansion: 0.3265

3D Ising model: Relative difference is less than $10^{-5}$

# Critical Temperature of the 3D Ising Model

<table>
<thead>
<tr>
<th>method</th>
<th>$T_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>HOTRG(D=16, from U)</td>
<td>4.511544</td>
</tr>
<tr>
<td>HOTRG(D=16, from M)</td>
<td>4.511546</td>
</tr>
<tr>
<td>Monte Carlo$^{37}$</td>
<td>4.511523</td>
</tr>
<tr>
<td>Monte Carlo$^{38}$</td>
<td>4.511525</td>
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<tr>
<td>Monte Carlo$^{39}$</td>
<td>4.511516</td>
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<tr>
<td>Monte Carlo$^{35}$</td>
<td>4.511528</td>
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<tr>
<td>Series expansion$^{40}$</td>
<td>4.511536</td>
</tr>
<tr>
<td>CTMRG$^{12}$</td>
<td>4.5788</td>
</tr>
<tr>
<td>TPVA$^{13}$</td>
<td>4.5704</td>
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<tr>
<td>CTMRG$^{14}$</td>
<td>4.5393</td>
</tr>
<tr>
<td>TPVA$^{16}$</td>
<td>4.554</td>
</tr>
<tr>
<td>Algebraic variation$^{41}$</td>
<td>4.547</td>
</tr>
</tbody>
</table>

Other RG methods:
Partial Order and Finite-Temperature Phase Transitions in Potts Models on Irregular Lattices

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Potts model

\[ H = J \sum_{<i,j>} \delta_{\sigma_i \sigma_j} \]

\[ i = 1, ..., q \]

Ferromagnetic: \( J < 0 \)
- 2d: \( q > 4 \) first order phase transition
- 3d: \( q > 2 \) first order phase transition

Antiferromagnetic: \( J > 0 \)
- \( q < q_c \) 1st/2nd phase transition at finite temperature
- \( q = q_c \) critical at 0K
- \( q > q_c \) no phase transition
Can $q_c > 4$ in certain lattices?

<table>
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<tr>
<th>Lattice</th>
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<th>$Q_c$</th>
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<tbody>
<tr>
<td>honeycomb</td>
<td>3</td>
<td>$&lt;3$</td>
</tr>
<tr>
<td>square</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>diced</td>
<td>4</td>
<td>$3 &lt; q_c &lt; 4$</td>
</tr>
<tr>
<td>kagome</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>triangular</td>
<td>6</td>
<td>4</td>
</tr>
<tr>
<td>union-jack</td>
<td>6</td>
<td>?</td>
</tr>
<tr>
<td>centered diced</td>
<td>6</td>
<td>?</td>
</tr>
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</table>
Irregular lattices

A lattice with two or more in-equivalent sub-lattices

Lave lattices
Phase transition with partial symmetry breaking

q=4 Potts Model on the UnionJack Lattice

Is there any phase transition?

\[ H = J \sum_{<i,j>} \delta_{\sigma_i \sigma_j} \]

\[ i = 1, \ldots, 4 \]

8 neighbors

4 neighbors
Full versus partial symmetry breaking

**full symmetry breaking**
Entropy = 0

**partial symmetry breaking**
Entropy is finite
Partial order and the ground state entropy

\[ S = \frac{N}{2} \ln 2 + 2 \times \frac{3N}{4} \ln \Box + \ldots \]

- If red or green sub-lattice is ordered, the ground states are \( \Box^{3N/4} \)-fold degenerate

- Both red and green sub-lattices are ordered, the ground states are \( 2^{N/2} \)-fold degenerate:

- All three sub-lattices are disordered
The red or green sublattice is ordered.

\[ \zeta = 1.7525 > 2^{2/3} \approx 1.5874 \]
Conjecture: there is a finite temperature phase transition

There is a partial symmetry breaking at 0K

q = 4 Potts model

There is a finite T phase transition with two singularities:
1. ordered and disordered states
2. $Z_2$ between green and red
Phase transition: specific heat jump
Finite-Temperature Phase Transition in a Class of Four-State Potts Antiferromagnets

Youjin Deng,1,* Yuan Huang,1,† Jesper Lykke Jacobsen,2,3,‡ Jesús Salas,4,§ and Alan D. Sokal5,6,‖

We argue that the four-state Potts antiferromagnet has a finite-temperature phase transition on any Eulerian plane triangulation in which one sublattice consists of vertices of degree 4. We furthermore predict the universality class of this transition. We then present transfer-matrix and Monte Carlo data confirming these predictions for the cases of the Union Jack and bisected hexagonal lattices.

\[ \text{(a)} \quad \text{(b)} \]
$q = 3$ Potts model on the Union–Jack lattice

There are two phase transitions:
1. Ordered phase I: all three sublattices ordered
2. Ordered phase II: red and green sublattices ordered
Partial order phase transition is ubiquitous on irregular lattices

Phase Transition in the Three-State Potts Antiferromagnet on the Diced Lattice

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3Gregorio Millán Institute, Universidad Carlos III de Madrid, 28911 Leganés, Spain
Partial order phase transition is ubiquitous on irregular lattices

coordination number 12, 6, 4 for sublattice red, blue and yellow

Centered Diced lattice
Partial order phase transition is ubiquitous on irregular lattices
Critical $q$ for the antiferromagnetic Potts model

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Summary

Tensor renormalization group provides an accurate and efficient numerical tool for studying 2D or 3D lattice models.

2D antiferromagnetic Potts model shows an ubiquitous partial order phase transition on irregular lattices.

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Zhiyuan Xie, Huihai

Mingpu Qin, Jing Chen, Jinwei Zhu
Green or Red Sub-lattice Magnetization

$q = 4$ Potts model on the Union-Jack lattice