Why Zhang’s Proposed ”Exact Solution” of the Three-Dimensional Ising Model is Wrong

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Abstract:
In 2007 Dr. Zhi-Dong Zhang conjectured a solution for the free energy and spontaneous magnetization per site of the three-dimensional Ising model. With Dr. Norman H. March he has advertised this solution as exact in several follow-up papers. We shall rigorously show that the conjectured solution is not correct, however. For this purpose we shall provide a detailed proof of the corresponding specialization of a general theorem of the 1960s, only using rather elementary mathematics. (See also arXiv:1209.0731 for further details.)
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In 2007 Zhi-Dong Zhang published the very long paper:


After being criticized in several comments, Zhang published many more papers mostly with Norman H. March. The last one contains all relevant references:


We shall see that there are many errors invalidating all claims.
Zhang’s results violate established series results, even in first nontrivial orders

To show this it is sufficient to restrict ourselves to the isotropic cubic Ising model \((J_1 = J_2 = J_3 = J, K = \beta J = J/k_B T)\). For that case Zhang expands his “putative” exact partition function per site in (A13) on page 5406 of [1] as

\[
Z^{1/N} = 2 \cosh^3 K \left[ 1 + \frac{7}{2} \kappa^2 + \frac{87}{8} \kappa^4 + \frac{3613}{48} \kappa^6 + \cdots \right], \quad \kappa = \tanh K.
\] (1)

This differs from the well-known high-temperature series given in (A12) of [1] as

\[
Z^{1/N} = 2 \cosh^3 K \left[ 1 + 0 \kappa^2 + 3 \kappa^4 + 22 \kappa^6 + \cdots \right].
\] (2)

Zhang then claims that both results are correct, the first for finite temperatures, the second only for an infinitesimal neighborhood of \(\beta \equiv 1/k_B T = 0\).

We shall rigorously show that the first result (1) is wrong and that series (2) has a finite radius of convergence. Our proof shall be rather elementary, compared to the proofs given in the 1960s.
From Eq. (103) on page 5342 of [1] we obtain the “putative” low-temperature expansion of the spontaneous magnetization,

\[ I = 1 - 6x^8 - 12x^{10} - 18x^{12} + \cdots, \quad x = e^{-2K}. \] (3)

Some coefficients of the well-known expansion are listed in Table 2 on page 5380 of [1], implying

\[ I = 1 - 2x^6 - 12x^{10} + 14x^{12} - \cdots. \] (4)

The textbook derivation starts with a \( d \)-dimensional hypercubic lattice of \( N \) sites, each site having \( 2d \) neighbors. There is one state with all spins up and energy \( E_+ \), \( N \) states with only one spin down and energy \( E_+ + 4dJ \), etc. Thus,

\[ I = \frac{Z_\sigma}{Z} = \frac{1 + (N - 2)x^{2d} + \cdots}{1 + Nx^{2d} + \cdots} = 1 - 2x^{2d} + \cdots, \quad 2d = 6. \] (5)

Only if the down spin is at the position of the spin operator we get a minus sign, or \( N - 2 = (N - 1)(+1) + (-1) \).

Again, Zhang’s result (3) is clearly wrong, displaying \( d = 4 \) behavior.
It has been shown in the 1960s that the high-temperature series of $\beta f = \lim Z^{1/N}$ and all correlation functions on the cubic lattice have finite nonzero radius of convergence. By duality with the Ising model with 4-spin interactions on all faces of the cubic lattice and at high temperatures, the low-temperature series for the spontaneous magnetization $I$ should also converge.

One cannot have two different expansions as given by Zhang.

To get out of this dilemma, Zhang (on pages 5381, 5383 and 5394 of [1]) comes with the mathematically absurd claim that the old series are asymptotic with zero radius of convergence, whereas his “putative solution” is analytic with finite radius of convergence. On pages 12 and 13 of [2] (and elsewhere) he claims that there is an (essential?) singularity at $\beta = h = 0$ and that this is due to the zeros of $Z^{-1}$ and that Perk “went on perpetrating the fraud, discussing the singularity of” $\beta f$, not $f$. 
Irrelevant pole of $f$

Perk proved (details later in this talk) that $\beta f$ has an absolutely convergent series, uniformly convergent in the thermodynamic limit, so that

$$\beta f = \sum_{i=0}^{\infty} a_i \beta^i, \quad |\beta| < r.$$  \hfill (7)

Therefore,

$$f = \frac{a_0}{\beta} + \sum_{i=1}^{\infty} a_i \beta^{i-1}, \quad 0 < |\beta| < r,$$  \hfill (8)

is a convergent Laurent series, totally equivalent to (7).

The pole at $\beta = 0$ has no significance, as $\beta f$ is the relevant quantity from statistical mechanics point of view, entering the normalization $Z = e^{-N\beta f}$ for the Boltzmann–Gibbs canonical distribution.

Also, note that in [1] Zhang expanded $Z^{1/N} = e^{-\beta f}$, not $f$. 

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Zeros of $1/Z$ are irrelevant

Zhang and March repeatedly claim the importance of the zeros of $Z^{-1}$, with $Z = z^N$ the total partition function. However, unlike the complex Yang–Lee zeros of $Z$, the zeros of $Z^{-1}$ are irrelevant:

For a finite number $N$ of sites, $Z$ is a finite Laurent polynomial in $e^K$ and only can become infinite when $\text{Re} \, K = \pm \infty$, i.e. zero-temperature type limits.

For the infinite system, $N \to \infty$, and finite $K$, the infinity of $Z$ should be seen as just a manifestation of the thermodynamic limit, in which $z = Z^{1/N}$ remains finite.

One can easily see that $\beta f < 0$ for $K$ real, so that $Z = e^{-N\beta f} = \infty$ for all real $K$ when $N = \infty$. There is nothing special about the $N = \infty$ zeros of $Z^{-1}$.

Hence, Zhang and March cannot claim that the zeros of $Z^{-1}$ play any role. They are there the same way for the one-dimensional Ising model, $Z \approx (2 \cosh K)^N$, and free Ising spins in field $B$, $Z = (2 \cosh \beta B)^N$, for $N \to \infty$. 
Strategy: Analyticity of the correlation functions and their thermodynamic limits

We first prove that the correlation functions of the Ising model on the periodic cubic lattice of size $N = n^3$, have high-temperature series absolutely convergent when $|\beta J| < r_0$, independent of $n$. The corresponding analyticity statement for $\beta f_N$ then follows using the elementary identity \( \frac{\partial (\beta f_N)}{\partial \beta} = u_N = 3J \langle \sigma \sigma' \rangle_N \), with $\sigma$ and $\sigma'$ nearest-neighbor spins.

As more and more coefficients become independent of $n$, the analyticity then carries over in the thermodynamic limit $N \to \infty$.

**Remark:** From a well-known corollary to Bogoliubov’s inequality, we have

\[
| \log \text{Tr } e^A - \log \text{Tr } e^B | \leq \| A - B \|, \quad \text{for } A \text{ and } B \text{ hermitian.} \tag{9}
\]

Using this one can show that, for $\beta J$ real and bounded, the free energy per site $f_N$ converges uniformly to a limit $f$ as the system size $N$ becomes infinite in the sense of van Hove. (Ratio of number of boundary terms and volume tending to zero.) But this does not yet show analyticity.
Laurent Polynomial Lemma

The partition function $Z_N$ is a Laurent polynomial in $e^{\beta J}$, i.e. polynomial in $e^{\beta J}$ and $e^{-\beta J}$. Therefore, $\beta f_N$ is singular only for the zeros of this Laurent polynomial and for some cases with $|e^{\pm \beta J}| = \infty$. As $Z_N$ is a sum of positive terms for real $\beta J$, $Z_N$ cannot have zeros on the real $\beta J$-axis.

It follows that all correlation functions $\langle \prod_l \sigma_l \rangle_N$ are meromorphic functions with poles only at the zeros of $Z_N$.

Hence,

$$\langle \prod_l \sigma_l \rangle_N = \sum_{i=1}^{\infty} c_i (\beta J)^i, \quad |c_i| < C_N r^{-i}, \quad (10)$$

with $r$ the absolute value of the zero closest to $\beta J = 0$ and $C_N$ some positive constant.
Lemma (M. Suzuki, 1965) for $B = 0$

\[
\left\langle \prod_{i=1}^{m} \sigma_{j_i} \right\rangle_N = \frac{1}{m} \sum_{k=1}^{m} \left\langle \left( \prod_{\substack{i=1 \atop i \neq k}}^{m} \sigma_{j_i} \right) \tanh \left( \beta J \sum_{l \text{ nn } j_k} \sigma_l \right) \right\rangle_N,
\]

(11)

where $j_1, \ldots, j_m$ are the labels of $m$ spins and $l$ runs through the labels of the six spins that are nearest neighbors of $\sigma_{j_k}$. (The lemma also holds without averaging over $k$.)

The proof simply follows summing over spin $\sigma_{j_k}$ in the numerator of the expectation value, i.e.,

\[
\sum_{\sigma_{j_k} = \pm 1} \sigma_{j_k} e^{\beta J \sum_{l \text{ nn } j_k} \sigma_{j_k} \sigma_l} = \tanh \left( \beta J \sum_{l \text{ nn } j_k} \sigma_l \right) \sum_{\sigma_{j_k} = \pm 1} e^{\beta J \sum_{l \text{ nn } j_k} \sigma_{j_k} \sigma_l}.
\]

(12)

Expanding tanh Lemma

\[
\tanh \left( \beta J \sum_{l=1}^{6} \sigma_l \right) = a_1 \sum_{(6)} \sigma_l + a_3 \sum_{(20)} \sigma_l \sigma_2 \sigma_3 + a_5 \sum_{(6)} \sigma_l \sigma_2 \sigma_3 \sigma_4 \sigma_5,
\]

(13)

where the sums are over the 6, 20, or 6 choices of choosing 1, 3, or 5 spins from the given \( \sigma_1, \ldots, \sigma_6 \). It is easy to check that the coefficients \( a_i \) are

\[
\begin{align*}
a_1 &= \frac{t(1 + 16t^2 + 46t^4 + 16t^6 + t^8)}{(1 + t^2)(1 + 6t^2 + t^4)(1 + 14t^2 + t^4)}, \\
a_3 &= \frac{-2t^3}{(1 + t^2)(1 + 14t^2 + t^4)}, \\
a_5 &= \frac{16t^5}{(1 + t^2)(1 + 6t^2 + t^4)(1 + 14t^2 + t^4)}, \\
t &= \tanh(\beta J).
\end{align*}
\]

(14)

The poles of the \( a_i \) are at \( t = \pm i, t = \pm (\sqrt{2} \pm 1)i, \) and \( t = \pm (\sqrt{3} \pm 2)i \). It can also be verified, e.g. expanding the \( a_i \) in partial fractions, that the series expansions of the \( a_i \) in terms of the odd powers of \( t \) alternate in sign and converge absolutely as long as \( |\beta J| < \arctan(2 - \sqrt{3}) = \pi/12 \).
Proof of Expanding tanh Lemma

Clearly, \( \tanh(\beta J \sum_{l=1}^{6} \sigma_l) \) can be expanded as done. Replacing all six spins, \( \sigma_l \) by \(-\sigma_l\), shows that no terms with an even number of spins occur. Also, permutation symmetry allows only three different coefficients.

Multiplying with one, three, or five spins \( \sigma_l \) and then summing over all \( 2^6 = 64 \) spin states, one finds the coefficients \( a_1, a_3, \) and \( a_5 \), which can the be expanded in partial fraction expansions,

\[
a_{1,5} = \frac{1}{24} \left( \frac{p_1 t}{1 + (p_1 t)^2} + \frac{p_2 t}{1 + (p_2 t)^2} \right) \pm \frac{\sqrt{2}}{8} \left( \frac{p_3 t}{1 + (p_3 t)^2} + \frac{p_4 t}{1 + (p_4 t)^2} \right) \\
+ \frac{1}{3} \frac{p_5 t}{1 + (p_5 t)^2},
\]

\[
a_3 = \frac{1}{24} \left( \frac{p_1 t}{1 + (p_1 t)^2} + \frac{p_2 t}{1 + (p_2 t)^2} \right) - \frac{1}{6} \frac{p_5 t}{1 + (p_5 t)^2}, \tag{15}
\]

\[
p_{1,2} = 2 \pm \sqrt{3}, \quad p_{3,4} = \sqrt{2} \pm 1, \quad p_5 = 1, \quad (p_1 p_2 = p_3 p_4 = 1). \tag{16}
\]

The remaining statements of the lemma follow from these expansions.
Uniform convergence for finite $N$

Expanding the tanh Suzuki’s lemma replaces any even correlation by a linear combination of even correlations, with coefficients given by the $a_i$. As, for fixed $|t| \equiv x$, each $|a_i|$ is maximal for imaginary $t = ix$, we find

$$s \equiv 6|a_1| + 20|a_3| + 6|a_5| \leq \frac{2x(3 - x^2)(1 - 3x^2)}{(1 - x^2)(1 - 14x^2 + x^4)}.$$  \hspace{1cm} (17)

It easily shown that the sum of the absolute values of all coefficients $s < 1$ for

$$|t| < (\sqrt{3} - \sqrt{2})(\sqrt{2} - 1) = 0.131652497\cdots,$$  or

$$|\beta J| < \arctan[(\sqrt{3} - \sqrt{2})(\sqrt{2} - 1)] = 0.130899693\cdots.$$  \hspace{1cm} (18)

This determines a lower bound on the radius of convergence:

Repeat applying Suzuki’s lemma on all correlations that are not $\langle1\rangle = 1$. After $I$ iterations, a given even correlation is than split into an explicitly given sum bounded by $s + s^2 + \cdots + s^I$ and a remainder sum with correlations left for further iteration, higher order in $t$ or $\beta J$. As we saw that each correlation can have at most poles given by the complex zeros of $Z$, we conclude that the remainder is bounded by $s^{I+1} + s^{I+2} + \cdots$. 

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Uniform convergence for infinite $N$

**Theorem:** $\langle \prod_{i=1}^{m} \sigma_{j_i} \rangle_N$ converges to a unique limit as $N \to \infty$ for $|t| < 2 - \sqrt{3}$, defined by its series, as more and more coefficients become independent of $N$ with increasing $N$ and the remainder tends to zero uniformly.

The above iteration process generates new correlations with the range of the positions $j_i$ of the spins extended by one in a given direction. As long as we do not go around a cycle (periodic boundary condition) of the 3-torus, we do not notice any $N$-dependence.

More precisely: Let $d$ be the largest edge of the minimal parallelepiped containing all sites $j_1, \ldots, j_m$. Then the coefficient of $t^k$ with $k < n - d$ for the lattice with $N = n^3$ sites equals the corresponding coefficient for larger $N$, including the one for $N = \infty$. It takes at least $n - d$ iteration steps to notice the finite size of the lattice.
Theorem for reduced free energy and its thermodynamic limit

The reduced free energy $\beta f_N$ for arbitrary $N$ and its thermodynamic limit $\beta f$ are analytic in $\beta J$ for sufficiently high temperatures. They have series expansions in $t$ or $\beta J$ with radius of convergence bounded below by (suzidn) and uniformly convergent for all $N$ including $N = \infty$. The first $n-1$ coefficients of these series for $N = n^3$ equal their limiting values for $N = \infty$.

Proof: To prove analyticity of $\beta f$ in terms of $\beta$ at $\beta = 0$ it suffices to study the internal energy per site or the nearest-neighbor pair correlation function, as

$$u_N = \frac{1}{N} \langle \mathcal{H}_N \rangle_N = \frac{\partial (\beta f_N)}{\partial \beta} = -3J\langle \sigma_{0,0,0}\sigma_{1,0,0} \rangle_N.$$  \hfill (19)

Here $\sigma_{0,0,0}$ and $\sigma_{1,0,0}$ can be any other pair of neighboring spins. The proof then follows from the theorem for correlation functions and integrating the series for $u_N$, using $Z_N|_{\beta=0} = 2^N$, implying $\lim_{\beta \to 0} \beta f_N = -\log 2$. 

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The first nontrivial coefficient

Apply Suzuki’s lemma once to $\langle \sigma_{0,0,0}\sigma_{1,0,0} \rangle$, we find

$$\langle \sigma_{0,0,0}\sigma_{1,0,0} \rangle = a_1 \langle 1 \rangle + O(t^2) = t + O(t^2) = \beta J + O(\beta^2).$$  \hspace{1cm} (20)

Hence,

$$\beta f = -\log 2 - \int_0^\beta 3J\langle \sigma_{0,0,0}\sigma_{1,0,0} \rangle d\beta = -\log 2 - \frac{3}{2} (\beta J)^2 + O(\beta^3).$$  \hspace{1cm} (21)

This and the next few terms so obtained agree with the usual series expansion (2), but disagree with Zhang’s “putative” exact result.

The only possible conclusion is that the conjectured answers of Zhang [1,2] are wrong and we shall next see more reasons why.
No obvious choice of weight functions

One problem with conjecture 2 [1,2] is that there is no obvious choice for the weight functions. In (49) on page 5325 of [1] Zhang writes

\[ N^{-1} \ln Z = \ln 2 + \frac{1}{2(2\pi)^4} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \ln \left[ \cosh 2K \cosh 2(K' + K'' + K''') \\
- \sinh 2K \cos \omega' - \sinh 2(K' + K'' + K''') (w_x \cos \omega_x \right.
+ w_y \cos \omega_y + w_z \cos \omega_z) \left] \, d\omega' \, d\omega_x \, d\omega_y \, d\omega_z, \right. \]

with \( K = K' = K'' = K''' \) for the isotropic cubic lattice.

Even though this is a result of flawed assumptions, it is an integral transform that could give the right answer with a suitable choice of weight functions \( w_x, w_y, w_z \). Zhang’s wrong “putative” result comes from the choice \( w_x = 1, w_y = w_z = 0 \). In appendix A of [1] Zhang reversely engineers an other truncated series choice derived from known coefficients of the Domb–Sykes high-temperature series. There is no more information than is in the limited series results provided by others, so that this is not an exact result.
Conjecture 1 is manifestly wrong

The original paper [1] has an error in the application of the Jordan–Wigner transformation pointed out by Perk. This error has only been corrected explicitly in [2], which makes it easy to pinpoint the error with Conjecture 1:

It is well-known that in the spinor representation of the orthogonal groups each element $g$ can be written as a “fermionic Gaussian” of the form

$$g = \exp \left( \frac{1}{2} \sum_i \sum_j A_{ij} \Gamma_i \Gamma_j \right), \quad A_{ji} = -A_{ij},$$

with Clifford algebra elements satisfying $\Gamma_i \Gamma_j + \Gamma_j \Gamma_i = 2\delta_{ij}$ and antisymmetric complex coefficients $A_{ij}$. The spinor representation has been used in the Ising context first by Kaufman in 1949.

The closure property of Lie groups says that any product or inverse of elements of this form is again of the same fermionic Gaussian form.
Remark: Lie group and Lie algebra

The group elements $g$ act on the $\Gamma$’s as

$$\Gamma_k \longrightarrow g \Gamma_k g^{-1}.$$  \hspace{1cm} (24)

Choosing infinitesimal

$$g = 1 + \frac{\varepsilon}{2} \sum_i \sum_j A_{ij} \Gamma_i \Gamma_j + O(\varepsilon^2),$$  \hspace{1cm} (25)

we find from (24) in $O(\varepsilon^2)$, that the corresponding Lie algebra action is

$$\left[ \frac{1}{2} \sum_i \sum_j A_{ij} \Gamma_i \Gamma_j , \Gamma_k \right] = \sum_i A_{ik} \Gamma_i,$$  \hspace{1cm} (26)

showing that the infinitesimal action is through multiplication by antisymmetric matrices, the generators of rotations.

Therefore, the $g$’s indeed form a representation of a rotation group.
Transfer matrix

It is well-known that the free energy per site in the thermodynamic limit does not depend on boundary conditions. Therefore, the Hamiltonian (1) in [1] can be rewritten using the scww boundary conditions of Kramers and Wannier as

$$-\beta \hat{H} = \sum_{\tau=1}^{n} \sum_{j=1}^{ml} \left[ K s_j^{(\tau)} s_j^{(\tau+1)} + K' s_j^{(\tau)} s_{j+1}^{(\tau)} + K'' s_j^{(\tau)} s_{j+m}^{(\tau)} \right]. \quad (27)$$

For this purpose we have made the change of notation

$$s_{\rho,\delta}^{(\tau)} \equiv s_j^{(\tau)}, \quad j \equiv \rho + (m - 1)\delta, \quad \sum_{\rho=1}^{m} \sum_{\delta=1}^{l} = \sum_{j=1}^{ml}, \quad (28)$$

where $\tau = 1, \ldots, n; \quad \rho = 1, \ldots, m; \quad \delta = 1, \ldots, l.$

This leads to the transfer matrix $T = V_3 V_2 V_1$ in [2], with

\[ V_3 = \exp \left( -iK'' \sum_{j=1}^{ml} \frac{J+m-1}{\prod_{k=j+1}^{2k-1,2k} i} \left[ \Gamma_{2j} \frac{\Gamma_2}{\Gamma_{2j+2m-1}} \right] \right), \]  
(29)

\[ V_2 = \exp \left( -iK' \sum_{j=1}^{ml} \Gamma_{2j} \Gamma_{2j+1} \right), \quad V_1 = \exp \left( iK^* \sum_{j=1}^{ml} \Gamma_{2j-1} \Gamma_{2j} \right), \]  
(30)

compare (15), (16) and (17) of [2], with \( n \) replaced by \( m \). Clearly, (29) is not of the fermionic Gaussian form and, therefore, not an element of the group.

At this point, Zhang introduces a fourth dimension, stacking \( o \) copies of the model. Without changing the free energy per site, one can connect the copies to give (29) and (30) with the upper bounds of the sums \( ml \) replaced by \( mlo \).

Next, Zhang made the absurd conjecture that multiplying \( V_3'V_2'V_1' \) so obtained by

\[ V_4' = \exp \left\{ -iK'''' \sum_{j=1}^{mlo} \Gamma_{2j} \Gamma_{2j+1} \right\}, \quad K'''' = \frac{K'K''}{K}, \]

as given in (18) and (19) in [2], miraculously produces a rotation group element. The argumentation in [1] for the form of \( K'''' \) also makes no sense.
Appendix: Other issues

Zhang and March also falsely claim that setting $\beta = 1$ in 1960’s references is a loss of generality, losing $T = \infty$. On the contrary, as $\beta f$ is only a function of $K = \beta J$, this is no problem. Having $J \equiv K$ and choosing a fixed new $\bar{J}$ and a new $\bar{\beta} = J/\bar{J} \neq 1$, we can write $J = K = \bar{\beta} \bar{J}$. Thus we recover the general case with both a fully variable $\beta$ (including $\beta = 0$) and a new $J$ (omitting the bars).

Next, Zhang and March fail to realize that $K_{\beta \phi'}(X, T)$ [in Phys. Lett. A 25 (1967), 493–494] vanishes for $\beta = 0$. The cited inequality does not fail for $\beta = 0$.

Finally, only in two dimensions is the conformal group infinite-dimensional, so that there is further inconsistency using Virasoro algebra in [2] and cited papers, beyond the fact that these papers build on an erroneous solution of the 3D Ising model. That Zhang and March write $\text{Re}|e^{i\phi_1}|$, the real part of a positive real number, is objectionable too.
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