Ising-like transitions in the $O(n)$ model on the square lattice

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Reference:
- Phys. Rev. E 83, 021115 (2011)
Background, $O(n)$ spin model and $O(n)$ loop model

Exact results of $O(n)$ loop model in 2D

Transfer matrix analysis

Results for phase diagram and critical properties

Summary
O(n) spin and loop models on the square lattice

Model of $n$-component spins on the square lattice with $O(n)$-symmetric couplings Blöte and Nienhuis 1989, Nienhuis 1990

$$Z_{\text{spin}} = \int \left[ \prod_{i} d\vec{s}_i \right] \prod_{V} \left\{ 1 + x(\vec{s}_1 \cdot \vec{s}_2 + \vec{s}_2 \cdot \vec{s}_3 + \vec{s}_3 \cdot \vec{s}_4 + \vec{s}_4 \cdot \vec{s}_1) + y(\vec{s}_1 \cdot \vec{s}_3 + \vec{s}_2 \cdot \vec{s}_4) + z[(\vec{s}_1 \cdot \vec{s}_2)(\vec{s}_3 \cdot \vec{s}_4) + (\vec{s}_2 \cdot \vec{s}_3)(\vec{s}_4 \cdot \vec{s}_1)] \right\}$$
O(n) spin and loop models on the square lattice

Model of \( n \)-component spins on the square lattice with \( O(n) \)-symmetric couplings \( \text{Bl"ote and Nienhuis 1989, Nienhuis 1990} \)

\[
Z_{\text{spin}} = \int \left[ \prod_{i} d\vec{s}_i \right] \prod_{V} \left\{ 1 + x (\vec{s}_1 \cdot \vec{s}_2 + \vec{s}_2 \cdot \vec{s}_3 + \vec{s}_3 \cdot \vec{s}_4 + \vec{s}_4 \cdot \vec{s}_1) + y (\vec{s}_1 \cdot \vec{s}_3 + \vec{s}_2 \cdot \vec{s}_4) + z [(\vec{s}_1 \cdot \vec{s}_2)(\vec{s}_3 \cdot \vec{s}_4) + (\vec{s}_2 \cdot \vec{s}_3)(\vec{s}_4 \cdot \vec{s}_1)] \right\}
\]

This special choice allows a mapping on a noninteracting loop model:

\[
Z_{\text{spin}} = Z_{\text{loop}} = \sum_{G} x^{N_x} y^{N_y} z^{N_z} n^{N_l}
\]

- spin dimension \( n \rightarrow \) loop weight, continuous variable
- no crossing bonds
- loop weight \( n \rightarrow \) long-range interactions if \( n \neq 1 \)
Exact results on the square lattice

There are 4 exactly solved branches, parametrized by $n$ Batchelor et al., PRL, 1989.

- Deduced $(x, z)$ phase diagram for $y \approx x$
  
  ▶ branch 1: spin ordering, dilute to dense phase: largest loops diverge
  ▶ branch 2: low-temperature branch, dense phase
  ▶ branch 4: Ising transition in dense phase
  ▶ branch 3: multicritical point where 3 transition lines merge

Blöte and Nienhuis, JPA, 1989; Guo, Blöte and Nienhuis, IJMP, 1999
Low-temperature branch

- branch 2: algebraically decay of correlation; **NOT long-range ordered** (except for $n=1$)

- Jacobsen *et al.* [PRL 2003]: $O(n)$ scalar field $\rightarrow$ loop model with crossings. The crossing leads to an ordered phase.


- Loop model we discussed: loops never cross each other, NOR connect to each other.
The effects of connecting and crossing perturbations

a loop model with crossing and cubic bonds

\[ Z_{\text{loop}} = \sum_G n^l x^{N_x} y^{N_y} z^{N_z} r^{N_r} c^{N_c} \]

By means of TM calculation and finite-size scaling, we found

- Crossing bonds lead to **crossover to a different universality class of dense intersecting loop models**
- Cubic bonds lead to **crossover to a cubic long range ordered phase**
- Our numerical results are coincident with Coulomb gas predictions

Guo and Blöte, PRE, 83, 021115 (2011)
Why Ising transition in dense loop phase?

Put dual Ising spins on faces such that:
- neighbor spins are different if separated by loop;
- neighbor spins are equal if not separated by loop.

The loops represent interfaces of the dual Ising model.

No AF Ising order

z-type vertices dominate $\rightarrow$

AF Ising ordered
What happens for $n > 2$?
Is the $n < 2$ phase diagram robust?
Is the case $y = 0$ special?
Remaining questions

1. What happens for $n > 2$?
2. Is the $n < 2$ phase diagram robust?
3. Is the case $y = 0$ special?

For simplicity: we study systems characterized by single parameter $x$

- case 1: $y = x, z = x^2$
- case 2: $y = 0, z = x^2$

No exact solution, use numerical methods

- transfer matrix analysis
Transfer matrix procedure

Wrap $O(n)$ model on cylinder with circumference $L$
Construct transfer matrix $T$, find largest eigenvalue $\Lambda_0$
Then, the free energy density $f(L) = \ln(\Lambda_0)$ scales as

$$f(L) \simeq f(\infty) + \frac{\pi c}{6L^2}$$

where $c$ is the conformal anomaly Cardy, D&L, Vol.11.

Correlation length $\xi_k(L)$ related to subleading eigenvalue $\Lambda_k$:

$$\xi_k^{-1}(L) = \ln\left(\frac{\Lambda_0}{\Lambda_k}\right)$$

At critical point, $\xi_k(L)$ scales as Cardy, D&L, Vol.11

$$X_k \simeq \frac{L}{2\pi \xi_k(L)}$$

Thus, transfer matrix calculations and finite-size scaling can be used to find critical points, universality classes.
Results: case $x = y$

$-2 < n \leq 2$, we find

- O($n$) critical line branch 1
- O($n$) low-temp line branch 2
- Ising-like transition in the dense phase

Magnetic correlation function

$$g_m = \langle \vec{s}_0 \cdot \vec{s}_r \rangle$$

At critical point, correlation length scales as

$$X_h = \frac{L}{2\pi \xi_h^{-1}}(L) = \frac{L}{2\pi} \ln(\Lambda_0/\Lambda_h)$$

$$X_h^{(LTI)} = X_h^{(LT)} + 1/8, \quad c^{(LTI)} = c^{(LT)} + 1/2$$
Results: case $x = y$

$W = \frac{1}{x(n+10)^{1/4}}$ temperature-like quantity

- $n \gg 2$, hard-square lattice gas
  - high temperature $\rightarrow$ solid pattern!

- an Ising-like transition in between.
  - consider density correlation function $g_s(r)$, with correlation length $\xi_s^{-1}(L) = \ln(\Lambda_0/|\Lambda_s|)$, $\Lambda_s$ belongs to eigenstate of $T$ that changes sign under elementary shift.
  - critical line determined by $X_s = 1/8$
  - we find $c = 0.5$

- Dense phase, wrapping loops destroy the checkerboard pattern
- AF Ising ordered
- $n \gg 2$, hard-square lattice gas
- high temperature $\rightarrow$ solid pattern!
- $x, z$ small: few small loops, randomly distributed
- $x, z$ large: small loops covering half faces, checkerboard pattern
- an Ising-like transition in between.
- consider density correlation function $g_s(r)$, with correlation length $\xi_s^{-1}(L) = \ln(\Lambda_0/|\Lambda_s|)$, $\Lambda_s$ belongs to eigenstate of $T$ that changes sign under elementary shift.
- critical line determined by $X_s = 1/8$
- we find $c = 0.5$
**Results:** case $x = y$

- $n \gg 2$, hard-square lattice gas high temperature $\rightarrow$ solid pattern!

- Dense phase, wrapping (diverge) loops destroy the checkerboard pattern
- AF Ising ordered

\[ W = \frac{1}{x(n+10)^{1/4}} \]  

temperature-like quantity

- $n < 2$, transition can not be seen from $\xi_s$
- $n > 2$, the lattice gas transition can not be seen from $\xi_h$, since there is no long magnetic correlation, or, no infinitely long loops
Results: case $y = 0$

\[ W = \frac{1}{x(n+10)^{1/4}} \text{ temperature-like} \]

- $n >> 2$, lattice-gas transitions, Ising universality class
- Critical manifold continues into $n \leq 2$, however
  - critical line determined by $\xi_h$ for $n \leq 1$, by $\xi_s$ for $1 < n \leq 2$
  - critical properties do not belong to known universality classes
Results: case $y = 0$

- $n \gg 2$, lattice-gas transitions, Ising universality class
- Critical manifold continues into $n \leq 2$, however
  - critical line determined by $\xi_h$ for $n \leq 1$, by $\xi_s$ for $1 < n \leq 2$
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\[ W = \frac{1}{x(n+10)^{1/4}} \] temperature-like

- an Ising-like degree of freedom frozen along each loop for $y = 0$.
- at the $y = 0 \, O(n)$ critical point, where the largest loop diverges, there might be ordering of loop color, allowing for different universal behavior?
Various transitions in square-lattice $O(n)$ models

- for $n > 2$ hard-square-like transition
- for $n < 2$ it depends on $y = x$ or $y = 0$
- $x = y$: hard-square line connects to Ising line in dense phase
- $y = 0$, phase diagram different, new type of transition

Thank You!